Mathematical Reviews

July, 1940 — Vol. 1, No. 7

TABLE OF CONTENTS

THE REPRINT SERVICE

The subscribers to MATHEMATICAL REVIEWS can obtain, either on microfilm or as photoprint, the complete text of any article of which an abstract is printed, except books, or material of which the reproduction is prohibited by copyright law. The microfilm requires some sort of reading machine or projector in which standard 35 mm. double perforated film can be viewed. Only *positive* prints are furnished; that is, the projected image appears as black words on a white background. Photoprints, on the other hand, are on sheets 10" by 14", and are similar to photostats. They can be read directly without the aid of any reading device. They also are furnished as positives, that is, black words on a white background.

Cost

The cost of microfilm is 2¢ per exposure, *each exposure* (except in the case of a very few unusually large journals) *containing two pages of text.*

Photoprints cost 16¢ per sheet, *each sheet* (except in the case of unusually large journals) *containing two pages of text.*

The minimum charge for any one order (but not for any one item when several items are ordered simultaneously) is 50¢.

How to Order

The process of ordering either microfilm or photoprint may be made clear by considering an example. Each abstract bears a serial number easily recognized by the fact that it is always preceded by the abbreviation "MF". A subscriber who wishes the text of item 18 would first note that the original article occupied pages 590 to 608. It therefore covers 19 pages, and will require ten exposures of microfilm, or ten sheets of photoprint for its reproduction. Hence the microfilm charge for this particular item would be 20¢, and the photoprint charge $1.60. The charges so computed apply on any order totaling 50¢ or more; if the total is less than 50¢, the charge for the entire order will be 50¢.

In ordering:

1. Give your name and address.
2. State the serial number of the article or articles you wish.
3. Say whether you wish microfilm or photoprint.
4. Enclose cash for the amount of the order, either in currency, stamps, or by check.
5. Mail to MATHEMATICAL REVIEWS, Brown University, Providence, Rhode Island.

In the case of large journals, only one page of which can be accommodated on a microfilm exposure or a photoprint sheet, you will be notified of the extra charge before the order is filled, unless you have indicated on your order that the reprint is desired even at the higher price.

Why Cash Must Accompany Order

Individual orders for either microfilm or photoprint are so small that if credit books were kept and bills rendered, the extra clerical expenses would be a large proportion of the total. By requiring payment in advance, these charges have been eliminated, and the price to the customer correspondingly reduced. However, libraries and institutions which make extensive use of the service may be granted the privilege of payment at stated intervals.

ONE-SIDE EDITION OF MATHEMATICAL REVIEWS

In order to supply persons interested in making card files for subject indices of their own fields, or to add bibliographical remarks to the REVIEWS in the future, there will be available an edition of MATHEMATICAL REVIEWS printed on only one side of the paper. This special edition may be obtained for an additional payment of $1.00 (that is, $14.00 for subscriptions, and $7.50 for members of sponsoring organizations). This edition will be folded but not stitched. Subscriptions for this edition should be entered as soon as possible since only a limited number can be printed. For the first four issues, only the two-side edition is available. A current regular subscription can be changed to the one-side subscription by informing MATHEMATICAL REVIEWS and paying $1.00.

MATHEMATICAL REVIEWS

Published monthly by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

THE AMERICAN MATHEMATICAL SOCIETY
THE MATHEMATICAL ASSOCIATION OF AMERICA
ACADEMIA NACIONAL DE CIENCIAS EXACTAS, FISICAS Y NATURALES DE LIMA
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879.

Mathematical Reviews

Vol. 1, No. 7 JULY, 1940 Pages 193-224

ALGEBRA

★**Barnard, S. and Child, J. M. Advanced Algebra.** Macmillan and Co., Ltd., London, 1939. x+280 pp. \$4.00.

This volume is a continuation of the authors' Higher Algebra and is intended for the mathematical specialist. Its scope has been determined by what is necessary in algebra for honors degrees at British universities. Some theorems and proofs are original with the authors, but they hesitate to claim them as new. *R. D. Carmichael* (Urbana, Ill.).

Equations, Polynomials

Specht, Wilhelm. Wurzelabschätzungen bei algebraischen Gleichungen. Jber. Deutsch. Math. Verein. 49, 179–190 (1940). [MF 1213]

Let $x^n+a_1x^{n-1}+\cdots+a_n=0$ be an algebraic equation of degree n with roots $\xi_1, \xi_2, \cdots, \xi_n$. The author determines bounds for the magnitude of the roots in terms of the magnitude of the coefficients of the equation. Let $S_r=|\xi_1|^r+\cdots+|\xi_n|^r$, r an integer. If $\alpha=\max|a_k|^{1/k}$, it is shown that $S_1<\alpha(n+\log n)$, and similar inequalities are obtained for S_r when $r>1$. Other inequalities are obtained by writing the coefficients of the equation as elements of a determinant and making use of known results for the characteristic roots of a determinant. For example, $S_2\leqq n-1+b$, where $b=|a_1-1|^2+|a_2-a_1|^2+\cdots+|a_n-a_{n-1}|^2+|a_n|^2$.
A. C. Schaeffer (Palo Alto, Calif.).

Anghelutza, Th. Sur une limite des modules des zéros des polynomes. Acad. Roum. Bull. Sect. Sci. 21, 211–213 (1939). [MF 1508]

A theorem belonging to the type introduced by Féjer [Sur la racine de moindre module d'une équation algébrique, C. R. Acad. Sci. Paris 145, 459–461 (1907)]. Féjer's result was generalized, among others, by Montel and E. van Vleck [for references see the paper]. The present paper contains the following extension of their results: Given (1) $a_0+a_1x+\cdots+a_nx^n=0$, of any degree. For any $p<n$, the first p coefficients $a_0, a_1, \cdots, a_{p-1}$ and, besides, one other coefficient $a_{p+m}(\neq 0)$, m any one of the values $0, 1, \cdots, n-p$, are fixed. Then at least p roots of (1) are in absolute value smaller than the positive root of

$$|a_{p+m}|x^{m+p}-C_{n-p+1}^{m+1}|a_{p-1}|x^{p-1}-C_{n-p+2}^{m+2}\cdot C_{m+1}^{1}|a_{p-2}|x^{p-2}$$
$$-\cdots-C_{n-1}^{m+p-1}\cdot C_{m+p-2}^{p-2}|a_1|x-C_{n}^{m+p}\cdot C_{m+p-1}^{p-1}|a_0|=0.$$

Proof is outlined. *A. J. Kempner* (Boulder, Colo.).

Obreschkoff, Nikola. Über algebraische Gleichungen, die nur Wurzeln mit negativen Realteilen besitzen. Math. Z. 45, 747–750 (1939). [MF 1411]

A short and elementary proof of Hurwitz's theorem giving in terms of a certain set of determinants necessary and sufficient conditions for all roots of a real algebraic equation to have their real parts less than 0. The interesting proof is based on Biehler's theorem: If all zeros of the polynomial $\varphi(x)+i\psi(x)$ lie on one side of the axis of reals, the real polynomials $\varphi(x)$, $\psi(x)$ have all of their zeros simple and real, and each set separates the other set; and conversely.
A. J. Kempner (Boulder, Colo.).

Poivert, Jules. Résolution algébrique d'une importante classe d'équations. Rev. Trimest. Canad. 26, 71–78 (1940). [MF 1682]

Given a system S of n equations in $n-1$ unknowns $y, z, \cdots$, and in n constants $a, b, c, \cdots$, we may by elimination establish a relation $F(a, b, c, \cdots)=0$ between the constants $a, b, c, \cdots$. By replacing one of the constants (for example, a) by an unknown x, the eliminant becomes an equation in implicit form, $F(x, b, c, \cdots)=0$. It may happen that this eliminant can also be obtained from S in explicit form, $x=f(b, c, \cdots)$. In this case the system S furnishes both an equation $F(x, b, c, \cdots)=0$ and its solution $x=f(b, c, \cdots)$. Application to S: (1) $y+z=x$; (2) $yz=b$; (3) $y^n+z^n=c$. From (1), (2), and substituting into (3):

$$F(x, b, c, \cdots)=\left[\frac{x}{2}+\left(\frac{x^2}{4}-b\right)^{\frac{1}{2}}\right]^n+\left[\frac{x}{2}-\left(\frac{x^2}{4}-b\right)^{\frac{1}{2}}\right]^n-c=0.$$

From (2), (3), and substituting into (1):

$$x=f(b, c, \cdots)=\left[\frac{c}{2}+\left(\frac{c^2}{4}-b^n\right)^{\frac{1}{2}}\right]^{1/n}+\left[\frac{c}{2}-\left(\frac{c^2}{4}-b^n\right)^{\frac{1}{2}}\right]^{1/n},$$

which is therefore a solution of the preceding equation.

The multiple-valuedness of the radicals in $f(b, c, \cdots)$ is not investigated. Connection with the trigonometric identity for $\cos\alpha/n$ in terms of $\cos\alpha$. Connection with Cardan's formula for the solution of the reduced cubic.
A. J. Kempner (Boulder, Colo.).

Komischke, A. Sur la résolution non-algébrique de l'équation générale du degré n. Revista Ci., Lima 41, 453–474 (1939). [MF 1642]

The first twelve pages lead up to the known series expansion $z=\alpha_1y+\alpha_2y^2+\cdots$ for one of the roots z of an equation $a_1z+a_2z^2+\cdots+a_nz^n=y$. The validity of the expansion is guaranteed by function-theoretic considerations. Use is made of Fejér's theorem: $a_0+a_1x^{\nu_1}+\cdots+a_kx^{\nu_k}=0$ has at least one root satisfying

$$|x|^{\nu_1}\leqq\binom{\nu_1+k-1}{k-1}\cdot\left|\frac{a_0}{a_1}\right|.$$

The coefficients of the series are determined by means of a certain differential operator. It is not clearly stated what is claimed for the paper with respect to novelty. The reading is rather difficult on account of poor typography and numer-

ous misprints. The last ten pages—"Some theorems on the series"—I have not been able to follow.

A. J. Kempner (Boulder, Colo.).

Tatuzawa, Tikao. Über die Irreduzibilität gewisser ganzzahliger Polynome. Proc. Imp. Acad., Tokyo **15**, 253–254 (1939). [MF 1143]

In Verschärfung eines Resultates von G. Pólya [Jber. Deutsch. Math. Verein. **28**, 31–40 (1919)] beweist der Verfasser zwei Irreduzibilitätskriterien, darunter: Nimmt ein ganzzahliges Polynom $P(x)$ vom Grade n an n voneinander verschiedenen ganzzahligen Stellen Werte an, die sämtlich von 0 verschieden und absolut genommen kleiner als $(2^{-n}(n-1)!)^{\frac{1}{2}}$ sind, so ist $P(x)$ irreduzibel (im rationalen Zahlkörper).

G. Pólya (Zürich).

Vessiot, Ernest. Sur une théorie nouvelle de la réductibilité des équations algébriques. C. R. Acad. Sci. Paris **210**, 159–161 (1940). [MF 1629]

Given the equation $F(x)=x^n+p_1x^{n-1}+\cdots+p_n=0$ with n distinct roots $x_1, x_2, \cdots, x_n$ and a domain of rationality Δ which contains the coefficients of $F(x)$. Let P denote the corresponding system of n equations: $\Sigma_1^n x_1x_2\cdots x_k=(-1)^k p_k$ with $k=1, 2, \cdots, n$. Then $F(x)$ is reducible in Δ if there can be found for system P at least one subsystem which is rational in Δ. (By a subsystem Q of system P is meant a system of equations in the unknowns $x_1, x_2, \cdots, x_n$ such that every solution of Q is a solution of P.) Now, there are $n!$ transformations T of degree $n-1$: $x'=\Sigma_1^n t_k x^{n-k}$, which effect permutations of the x_i and hence leave invariant both equation $F(x)=0$ and the corresponding system of equations P. The ensemble I of these $n!$ "auto-transformations" form a "pseudo-group, modulus $F(x)$," meaning that, if $x'=g(x)$ and $x'=h(x)$ are any two transformations in I, then $x'\equiv h(g(x))$, mod $F(x)$, is also a transformation in I, and that, if $x'=g(x)$ is any transformation in I, then in I can be found a transformation $x'=h(x)$ such that $h(g(x))\equiv x$, mod $F(x)$. The author finds in I a pseudo-subgroup K of transformations that leave invariant every rational subsystem Q of system P. All the solutions of Q can be obtained by applying the transformations in K to any one solution of Q. Finally, the Galois group for the equation $F(x)=0$ is merely the group of permutations on the x_i which are produced by the various transformations of K.

M. Marden (Milwaukee, Wis.).

Ore, Oystein. A note on the factorization of polynomials. Revista Ci., Lima **41**, 587–592 (1939). [MF 1649]

If $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ is a polynomial with integral rational coefficients and $g(x)=x^2+ax+b$ a quadratic factor, where a and b are rational integers, then b must divide a_n. The possible values of a are usually discussed on account of the condition that $g(x_0)$ divides $f(x_0)$ for every integral rational value of x_0. The author remarks that instead the condition can be used that the discriminant of $g(x)$ divides the discriminant of $f(x)$.

R. Brauer.

Foulkes, H. O. Canonical matrix roots of equations. Proc. London Math. Soc. (2) **46**, 155–173 (1940). [MF 1501]

Let $f(x)=0$ be an equation of degree n with coefficients in a field R of characteristic 0. A set of n matrices of the same order m, which satisfy the equation $f(x)=0$, are commutative with each other, and are such that their elementary symmetric functions are equal to the corresponding elementary symmetric functions of the algebraic roots of the given equation, is called a set of conjugate matrices. A. R. Richardson [Quart. J. Math. **7**, 256–270 (1936)] has shown that, if $f(x)$ is irreducible in R and the matrices of a conjugate set are required to be rational in R, the smallest possible order m is the order of the Galois group of the given equation. The matrices of such a rational conjugate set are transformed into each other by a group of matrices of the same order, isomorphic to the Galois group. The author chooses canonical forms for this transforming group and gives explicit constructions for the corresponding canonical matrix roots of quadratic, cubic and three types of quartic equations.

N. H. McCoy (Northampton, Mass.).

Linear Algebra

Oldenburger, Rufus. Higher dimensional determinants. Amer. Math. Monthly **47**, 25–33 (1940). [MF 1153]

This is a brief exposition of p-way determinants. Theorem 11 is known.

C. C. MacDuffee (Madison, Wis.).

Oldenburger, Rufus. Complete reducibility of forms. Bull. Amer. Math. Soc. **46**, 88–92 (1940). [MF 1252]

Let K be a field of characteristic not 2 or 3. A form F of degree n in n essential variables is completely reducible in K if and only if F is the Hessian of a cubic form which can be written as a linear combination of cubes of linear forms with coefficients in K. More detailed results are obtained for $n=3$.

C. C. MacDuffee (Madison, Wis.).

Rados, Gusztáv. Über einige von Hermiteschen abgeleitete Formen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **58**, 639–651 (1939). (Hungarian. German summary) [MF 1792]

Rados, Gusztáv. Eine elementare Herleitung der Kennzeichen definiter und semidefiniter Hermitescher Formen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **58**, 652–666 (1939). (Hungarian. German summary) [MF 1793]

Rados, Gusztáv. Über die Kronecker'sche Composition von Hermiteschen Formen. Math. Naturwiss. Anz. Ungar. Akad. Wiss. **58**, 667–672 (1939). (Hungarian. German summary) [MF 1794]

In these papers the author gives a number of theorems (partly known) concerning the behavior of forms connected with given Hermitian forms. The proofs are elementary.

O. Szász (Cincinnati, Ohio).

Rados, Gustav. Über die Unabhängigkeit der Bedingungsgleichungen zwischen den Koeffizienten unitärer Substitutionen. Acta Litt. Sci. Szeged **9**, 201–205 (1940). [MF 1222]

Turri, T. Sulle omografie quadrato di antiomografie. Rend. Sem. Fac. Sci. Univ. Cagliari **9**, 195–203 (1939). [MF 1908]

Si trovano le condizioni necessarie e sufficienti perchè la matrice di un'omografia sia matrice dell'omografia quadrato di un'antiomografia.

Author's summary.

Bottema, O. The classification of affine transformations. Nieuw Arch. Wiskde **20**, 184–191 (1940). (Dutch) [MF 1097]

Let $(y)=A\cdot(x)$ be a non-singular affine transformation, using homogeneous coordinates. Then the bottom line of the

square $(n+1)$-rowed matrix A consists of zeros only, except for a 1 in the right-hand corner. The north-west n-rowed submatrix A_∞ characterizes the transformation in the infinite. There are transformations T that make $A'=TAT^{-1}$ have a submatrix A_∞' of canonical form. The author constructs T as an affine transformation and so that the last column of A' contains, apart from the bottom 1, one 1 at most and zeros everywhere else. A' is determined by the elementary divisors of A and A_∞; for the existence of an affine transformation $A\rightarrow B$, therefore, $A\sim B$ and $A_\infty\sim B_\infty$ are sufficient (and necessary). Furthermore, there results a classification of affine transformations by means of the elementary divisors. (Since the canonical form A_∞' used is Jordan's, the classification applies to A's with complex constituents.) *A. E. Mayer* (London).

Petr, K. Rationale kanonische Form einer linearen Substitution. Časopis Pěst. Mat. Fys. 69, 9–22 (1940). (Czech. German summary) [MF 1936]

Der Verfasser gibt eine Überführung der Substitution $X_i=\sum a_{ik}x_k$, $i, k=1, \cdots, n$, in ihre rationale kanonische Form an. Er benutzt im wesentlichen den Weg von L. E. Dickson [Modern Algebraic Theories], führt aber einen Operator Δ ein, durch $\Delta x_i=\sum a_{ik}x_k$, welcher wesentliche Vereinfachungen gestattet.

Extract from author's summary.

Givens, Wallace. Factorization and signatures of Lorentz matrices. Bull. Amer. Math. Soc. 46, 81–85 (1940). [MF 1250]

Define $(v, x)=v^1x^1+\cdots+v^tx^t-v^{t+1}x^{t+1}-\cdots-v^nx^n$, where v and x are two vectors. A linear transformation L taking x into $\bar{x}$ is called Lorentz if $(\bar{x}, \bar{x})=(x, x)$. By T_v denote the Lorentz transformation with the equations

$$\bar{x}=x-2\frac{(v, x)}{(v, v)}v.$$

This paper gives elementary algebraic proofs first that all Lorentz transformations can be expressed as products of transformations of the type of T_v, and second that, for any Lorentz transformations L and M, $\sigma_\pm(L)\sigma_\pm(M)=\sigma_\pm(LM)$, where $\sigma_\pm(L)$ are the so-called temporal and spatial signatures of L, that is, the respective signs of the determinants of the $t\times t$ minor in the upper left hand corner of L and of its complimentary minor. These results are not new, but the simplicity of proof and its elementary character are new.

B. W. Jones (Ithaca, N. Y.).

Gruner, W. Einlagerung des regulären n-Simplex in den n-dimensionalen Würfel. Comment. Math. Helv. 12, 149–152 (1939–40). [MF 1047]

It has been conjectured that a square orthogonal matrix of order m, with all its elements ±1, can be found whenever m is divisible by 4. (This is easily seen to be impossible for other values of m, except 1 and 2.) By considering quadratic residues in Galois fields, Paley [J. Math. Phys. Mass. Inst. Tech. 12, 311–320 (1933)] found certain classes of suitable values of m, which together account for all multiples of 4 less than 92. The present author carries on that work, adding a new class of values; in fact, he shows that such a matrix can be constructed whenever $m=k^2$, where each of the numbers $k\pm1$ is a power of a prime. It was pointed out by Barrau [Nieuw Arch. Wiskde (2) 7, 250–270 (1907)] that an orthogonal matrix of ±1's provides coordinates for the vertices of an $(m-1)$-dimensional regular simplex inscribed in an $(m-1)$-dimensional hyper-cube. Hence, 17 and 19 being primes, the above result (with $k=18$) shows that the 324 vertıces of a 323-dimensional regular simplex occur among the 2^{323} vertices of the hyper-cube. (Smaller values of k merely give values of m which were already covered by the known formulae.) *H. S. M. Coxeter* (Toronto, Ont.).

Ledermann, Walter. On a problem concerning matrices with variable diagonal elements. Proc. Roy. Soc. Edinburgh 60, 1–17 (1940). [MF 1451]

Let R be a nonnegative definite symmetric matrix of order n, and let R_x be obtained from R by replacing its diagonal elements by variables $x_1, x_2, \cdots, x_n$. Such matrices occur in the statistical theories of Thurstone [The Vectors of Mind, Chicago, 1935] and Thomson [The Factorial Analysis of Human Ability, London, 1939]. It had been observed in many practical cases that values $x_1, x_2, \cdots, x_n$ such that R_x is nonnegative definite of minimum rank are such that the trace $x_1+x_2+\cdots+x_n$ is also minimal. It is shown that "in general" this is true, but exceptional cases occur. If R_x can be made to be of rank 1, necessary and sufficient conditions are obtained which characterize the exceptional cases, but if the rank of R_x always exceeds 1, only sufficient conditions are obtained. *C. C. MacDuffee* (Madison, Wis.).

Gurr, C. E. The expression of an infinite lower semi-matrix in terms of its idempotent and nilpotent elements. Proc. Edinburgh Math. Soc. (2) 6, 61–74 (1939). [MF 1517]

This paper expresses a certain type of infinite matrix A, or again a scalar function $g(A)$ of A, as an infinite series of scalar multiples of idempotent matrices A_i (that is, $A_i^2=A_i$), and so gives a result analogous to a well-known property of finite matrices. The matrix A is an infinite lower semi-matrix, so that $a_{ij}=0$ whenever $i<j$. The condition that $\sum_i|a_{ii}|^{-1}$ converges is a natural assumption and secures progress. The a_{ii} are the latent roots of A, and the case when equalities exist among these roots is examined. Idempotent are now supplemented by nilpotent matrices η_i. The exponential function $\exp A$ is discussed, also the uniqueness of the infinite product $\Pi_i(1+A/a_{ii})$. *H. W. Turnbull.*

Abstract Algebra

★Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I. Algebra und Zahlentheorie. Teil 1. A. Grundlagen. B. Algebra. Heft 5. Krull, Wolfgang. Allgemeine Modul-, Ring-, und Idealtheorie. 54 pp. B. G. Teubner, Leipzig, 1939.

This is a summary of the ideal theory of a commutative ring $\mathfrak{J}$ in the abstract and highly general form given to this study by the Noether school, starting from the Dedekind ideal theory for algebraic numbers and the Kronecker-Lasker-Macauley investigations of ideals of polynomials. The introductory sections deal with groups with operators, the algebra of ideals ($\mathfrak{a}$, $\mathfrak{b}$, etc.), prime ideals ($\mathfrak{p}$) and the general characterization of the isolated primary components of an ideal. The latter concept provides an illustration of the way in which a notion, like that of primary component, is historically first introduced in terms of the specific prob-

lem of the decomposition of algebraic manifolds, and later generalized to arbitrary rings.

The second chapter concerns 0-rings which satisfy a finite chain condition (all ascending chains of ideals have finite length). An 0-ring in which every prime ideal $\mathfrak{p}\neq 0$ is maximal is called a U-ring. The theory developed here is of an "additive" character, largely inspired by the special case of polynomial ideals. A preliminary topic is the study of "lengths" of ideals in a ring Q which is primary (that is, contains but a single prime ideal, which consists of all the nilpotent elements of Q). Such rings are significant because they include all the direct summand in the decomposition of the residue-class rings $\mathfrak{J}/\mathfrak{a}$ of an arbitrary U-ring $\mathfrak{J}$. On this basis, using the relation between direct sums and ideal products, one obtains in any U-ring $\mathfrak{J}$ the unique decomposition of an ideal $\mathfrak{a}$ into its isolated primary components. Krull leaves unmentioned the fact that this theorem has as a very special case the unique decomposition theorem for ideals in ordinary algebraic number fields.

For a ring $\mathfrak{R}$ which satisfies only the ascending chain condition one has the Noether theory of the representation of an ideal as a "shortest" intersection of primary ideals, and the Grell analysis of the relation between the ideals of $\mathfrak{R}$ and the "extended" ideals in the prime ideal quotient ring $\mathfrak{R}_{\mathfrak{p}}$. If $\mathfrak{R}$ is an integral domain, this $\mathfrak{R}_{\mathfrak{p}}$ consists of all quotients a/b with b not in $\mathfrak{p}$. These quotient rings have a single maximal prime ideal consisting of all non-units of $\mathfrak{R}_{\mathfrak{p}}$. A ring with the latter property is called a "place-ring," because this property is characteristic for the rings describing places (branches) of algebraic curves. For place rings Krull gives a dimension theory, depending on certain formal power series developments analogous to those used for p-adic numbers. A section is devoted to the Hilbert decomposition and inertial groups, and the abstract discriminant theory for a suitable finite extension $\mathfrak{J}'$ of an integrally closed ring $\mathfrak{J}$.

Chapter III on multiplicative ideal theory presents various extensions of the classical algebraic number concepts. There is Noether's theorem that the ideals of an integral domain $\mathfrak{J}$ have a unique decomposition into prime ideals if and only if $\mathfrak{J}$ is an integrally closed U-ring. Given such a decomposition, the set of all (fractional) ideals form a group. A more general decomposition theorem, valid in any integrally closed ring with an ascending chain condition, holds for v-ideals (according to van der Waerden and Artin, the v-ideal $\mathfrak{a}_v$ determined by $\mathfrak{a}$ consists of all the multiples of the common divisors of the elements of $\mathfrak{a}$). A closely related topic is Prüfer's criteria for "multiplication rings" (defined as those integral domains in which all ideals with finite bases form a multiplicative group).

The v-ideals of a domain $\mathfrak{J}$ are but one instance of special operations which have recently been shown important for the study of general rings. Another such is the a-ideal $\mathfrak{a}_a$, which consists of all elements of the quotient field K which are integrally dependent upon $\mathfrak{a}$. By the methods of Kronecker, these a-ideals can be used to define a certain "functional ring" $\mathfrak{M}$ lying in the extension $K(x)$ of the quotient field K by an indeterminate x. This ring $\mathfrak{M}$ can also be obtained from a system of valuation ideals belonging to a set of valuation rings $\mathfrak{B}_t$ whose intersection is the original ring $\mathfrak{J}$. In general, the possible functional rings $\mathfrak{M}$ are in one-to-one correspondence with certain systems of valuation ideals satisfying an ingenious finiteness condition due to Lorenzen. The report covers other pertinent ideal-theoretic applications of valuations: integrally closed rings as intersections of valuations rings, a proof of the theorem of Kronecker on the content of a product of two forms in terms of valuations, and the like. A section is devoted to an abstract formulation of Krull's topological theory of the ideals of infinite algebraic number fields.

The fourth chapter, on U-rings, is devoted largely to extensive generalization of the usual tools used to describe the relation of a ring $\mathfrak{J}'$ of integral algebraic numbers to a smaller ring $\mathfrak{J}$ (say to the ring of ordinary integers). Covered are the Schmidt and Artin conditions for the existence of a finite modul-basis for $\mathfrak{J}'$ over $\mathfrak{J}$, in the case of inseparable extensions, the decomposition of the conductor of $\mathfrak{J}$, theorems on norms and relative residue-class degrees, and a generalization of the Hensel-Ore description of the constitution of the different. An essential tool is the Weber theory of invertible ideals. Mention is made of some unpublished results of Grell on the description of all possible intermediate rings between $\mathfrak{J}$ and $\mathfrak{J}'$, in the case of algebraic number rings.

The present article differs considerably in content and arrangement from Krull's earlier Idealbericht [Ergebnisse der Mathematik, vol. 4, no. 3, 1935]. The inclusion of the recent developments due to Prüfer, Grell, Krull, Lorenzen and others has clarified some points. The style of the report is highly abstract, with almost no examples. All theorems are stated as for rings satisfying specified postulated properties. To fully comprehend their significance the reader must continually recall the pertinent special cases of the classical theory. For instance, the v-ideal theory is given in Artin's short form. This is doubtlessly elegant, but the reviewer thinks that the original van der Waerden description of v-ideals in terms of maximal prime ideals [Math. Ann. **101**, 293–308 (1929)] is so illuminating that it should be included.

One omission was noted: the report on the purely multiplicative theory of v-ideals in terms of semi-groups should include the relevant work of A. H. Clifford [Arithmetic and ideal theory of commutative semi-groups, Ann. of Math. **39**, 594–610 (1938)]. *S. MacLane* (Cambridge, Mass.).

✶Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I. Algebra und Zahlentheorie. Teil 1. A. Grundlagen. B. Algebra. Heft 5. Krull, Wolfgang. Theorie der Polynomideale und Eliminationstheorie. 53 pp. B. G. Teubner, Leipzig, 1939.

The elementary theory of polynomial ideals is treated in the first chapter of this report. Lasker's theorem on the representation of a polynomial ideal as an intersection of primary ideals is applied to the decomposition of an algebraic manifold into irreducible manifolds. The Hilbert-Netto theorem on the zeros of an ideal and the basic basis theorem of Hilbert are stated for polynomials with coefficients in a general ring P. The reader should here be reminded that the basis theorem applies only if the coefficient ring P has a unit. (W. F. Eberlein has recently remarked that the theorem fails to be true if the ring P without a unit is the ring of even integers.) Van der Waerden's theory of the general and special zeros of an ideal and the dimensions of these zeros is stated. An essential result is the agreement of the ideal-theoretic dimension of a prime ideal, defined in terms of chains of prime over-ideals, with the geometric concept of the dimension of the corresponding manifold, defined in terms of its parametric representation (that is, in terms of the transcendence degree of a suitable residue-class

ring). Krull explains how these dimension concepts apply also to rings of formal (or convergent) power series. He also formulates carefully the effect upon the dimensions of prime and primary ideals if the field of coefficients is subjected to an extension or if certain of the original variables x_i are regarded as parameters.

The second chapter is devoted to a thorough summary of methods of elimination theory. After a statement of the ideas of Kronecker and Hentzelt, Krull turns to the important problem of obtaining in a finite number of steps the explicit representation of an ideal $\mathfrak{a}$ as an intersection of primary ideals. By a theorem of Hentzelt this can be reduced to the problem of obtaining the prime ideals $\mathfrak{p}$ belonging to the given ideal $\mathfrak{a}$. The König-Macauley method of resolvents (or "norms") will obtain the minimal prime ideals containing $\mathfrak{a}$. To obtain the others, one can apply a suitable elimination process to the Hentzelt "ground ideals." These, in turn, can be explicitly computed by a finite but arduous process due to Hermann.

Subsequent sections cover the definition and properties of resultant systems for homogeneous equations and for equations homogeneous separately in several sets of variables, the Hurwitz theory of inertial forms, the Mertens resultant for certain unhomogeneous systems and the u-resultant so useful in van der Waerden's geometric investigations. The "irrational representations" of resultant covers various extensions of the formula $\Pi_i\Pi_j(\alpha_i-\beta_j)$ for the resultant of two monic polynomials with roots α_i and β_j, respectively.

A treatment of "relation-preserving specializations" is given as a basis for van der Waerden's multiplicity concept, which defines the multiplicity of a "special" solution of a problem in terms of the related "general" problem. It is remarked that this method is not really useful if there are special solutions whose multiplicity in this sense is zero, as is actually the case in certain investigations of absolute irreducibility (Noether). The multiplicity theorem is applied to the intersection of a manifold by linear spaces, to the general Bézout theorem, to algebraic correspondence between manifolds, to the foundation of the Schubert method of counting constants and to the recent Chow-van der Waerden concept of an algebraic system of algebraic manifolds [Math. Ann. **113**, 692–704 (1937)].

The third chapter concerns refinements of the polynomial ideal theory. The calculation of prime ideals in a finite number of steps, as discussed above, requires explicit computations for the intersection $\mathfrak{a}\cap\mathfrak{b}$ and the quotient $\mathfrak{a}:\mathfrak{b}$ of two ideals $\mathfrak{a}$ and $\mathfrak{b}$. This is done by Hermann's method in terms of theorems of the usual form about the explicit computation of solutions of homogeneous and nonhomogeneous linear equations over a polynomial ring. One of the intermediate problems involved is that of determining when a given polynomial belongs to a specified polynomial ideal. For practical purposes this problem can better be treated by the Lasker and Hentzelt theorems. These are essentially generalizations of the famous M. Noether "Fundamentaltheorem."

From the projective point of view the homogeneous ideals are important. For these, the report covers the Hilbert theorem on the finiteness of chains of syzygies, the Hilbert function of an ideal and its calculation, the inverse system of Macauley, its relation to results of Gröbner on irreducible primary components, as well as certain analogous properties of perfect ideals.

Krull next introduces certain ideals $s_r(\mathfrak{a})$ consisting of functional determinants formed from a given ideal $\mathfrak{a}$. The usual theorem that a polynomial $p(x)$ has a multiple root only if $p(x)$ and $p'(x)$ have a common factor can be considerably generalized in terms of these functional determinants, although the proof of this general theorem [p. 47] has apparently not been given anywhere. The ideals $s_r(\mathfrak{a})$ are also involved in the study of rational transformations.

A final section is devoted to the resolution of the singularities of algebraic manifolds. The infinitely-near neighboring points are mentioned, but not in terms of the recent arithmetic analysis due to Zariski [Amer. J. Math. **60**, 151–204 (1938)]. The report closes with a summary of Hensolt's posthumous paper [S.-B. Phys.-Med. Sozietät Erlangen, 1939] on Schmeidler's generalization of the "neighboring point" idea. It seems unfortunate that the report was closed so soon as to miss Zariski's elegant arithmetic resolution of the singularities of an algebraic surface [these Rev. **1**, 26 (1940)].

The material of this report has many geometric applications, but its emphasis is strongly algebraic, so that these interpretations are often unmentioned. For instance, the ideal theory of integral algebraic function rings [p. 7] could be applied to the characterization of subvarieties of a given manifold.

Mention is made of related work of Gröbner on differential equations. In this connection the reviewer would suggest the inclusion of the Ritt-Raudenbush theory of algebraic differential equations [Semicentennial Addresses of the American Mathematical Society, vol. 2, 1938, pp. 35–55]. The manifold and ideal theory here found is intellectually closely akin to the polynomial ideal theory.

S. Mac Lane (Cambridge, Mass.).

Dilworth, R. P. Note on complemented modular lattices. Bull. Amer. Math. Soc. **46**, 74–76 (1940). [MF 1248]

The author gives a brief proof of the decomposability of complemented modular lattices into projective geometries. His discussion is based on the properties of neutral elements (elements of the "center"). *G. Birkhoff.*

Ward, Morgan. The arithmetical properties of modular lattices. Revista Ci., Lima **41**, 593–603 (1939). [MF 1650]

After a preliminary exposition of two properties of modular lattices discovered by Kurosch [Rec. Math. (Moscou) [Mat. Sbornik] **42**, 613–616 (1935)] and Dilworth [the latter unpublished], the author states a condition which he shows to be sufficient as well as necessary, at least in the presence of the ascending chain condition. This is, that "if neither $c\geqq q$ nor $q\geqq c$, then q is a component of every element $d\neq q$ of the quotient lattice $q/q\cap c$, and q has at least one complement in d containing c." *G. Birkhoff* (Princeton, N. J.).

Maeda, Fumitomo. Ideals in a Boolean algebra with transfinite chain condition. J. Sci. Hirosima Univ. Ser. A **10**, 7–36 (1940). [MF 1791]

The author says a generalized Boolean algebra L satisfies the $\aleph$-chain condition if every ascending (descending) chain has power less than $\aleph$. [Cf. A. N. Milgram, Proc. Nat. Acad. Sci. U. S. A. **26**, 291–293 (1940); these Rev. **1**, 220. Especially p. 292, "lower inductive of power M."] He gives a necessary and sufficient condition on an ideal J for the quotient-algebra L/J to satisfy this condition. He applies this as a technical tool in investigating (1) the algebra of all measure functions on a $\aleph_1$-Boolean algebra and (2) his spectral theory of general Boolean alge-

bras of projections (generalized resolution of the identity). The latter corresponds to a recent construction of Wecken [Math. Annalen **116**, 422–455 (1939)]. *G. Birkhoff.*

Uzkow, A. I. Zur Idealtheorie der kommutativen Ringe. I. Rec. Math. (Moscou) [Mat. Sbornik] N.S. **5** (47), 513–520 (1939). (German. Russian summary) [MF 1342]

Van der Waerden [cf. Moderne Algebra, vol. 2, Chapter 14, §103] has given a theory of ideals of algebraically closed domains of integrity R. If $\mathfrak{a}$ is an ideal, the ideal $(\mathfrak{a}^{-1})^{-1}$ is denoted by $\mathfrak{a}^*$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be equivalent if $\mathfrak{a}^*=\mathfrak{b}^*$. A multiplication of classes of equivalent ideals can be defined, and the decomposition of classes into "prime classes" can be studied. The author considers an arbitrary domain of integrity R. Following Krull [Idealtheorie, Ergebnisse der Mathematik, vol. 4, no. 3, Berlin, 1935, p. 118], he assumes that to every ideal $\mathfrak{a}$ corresponds an ideal $\bar{\mathfrak{a}}$ such that (1) $\bar{\bar{\mathfrak{a}}}=\bar{\mathfrak{a}}$, (2) $\overline{\sum \mathfrak{a}_\nu}=\overline{\sum \bar{\mathfrak{a}}_\nu}$ for any arbitrary set of terms $\mathfrak{a}_\nu$, (3) $\overline{\mathfrak{a}\mathfrak{b}}=\overline{\bar{\mathfrak{a}}\bar{\mathfrak{b}}}$, (4) $\overline{(a)\mathfrak{b}}=(a)\bar{\mathfrak{b}}$ for every principal ideal (a). The necessary and sufficient conditions are given that van der Waerden's theory holds for R, provided that the equivalence of two ideals $\mathfrak{a}$ and $\mathfrak{b}$ is defined by $\bar{\mathfrak{a}}=\bar{\mathfrak{b}}$. These conditions are quite analogous to E. Noether's conditions for the validity of the ordinary theory of ideals. If the conditions are satisfied, then $\bar{\mathfrak{a}}=\mathfrak{a}^*=(\mathfrak{a}^{-1})^{-1}$, so that the equivalence definition becomes identical with van der Waerden's definition. *R. Brauer* (Toronto, Ont.).

Mori, Shinziro. Zerlegung der Hauptideale aus Polynomringen in minimale Primideale. III. J. Sci. Hirosima Univ. Ser. A **10**, 1–6 (1940). [MF 1790]

Let $\mathfrak{J}$ be an integral domain in which every principal ideal has a representation as a finite power product of prime ideals. Suppose that $\{x_i\}$ is an arbitrary set of independent transcendentals over $\mathfrak{J}$. The author shows that every principal ideal of the polynomial ring $\mathfrak{J}[\{x_i\}]$ is a finite power product of minimal prime ideals in $\mathfrak{J}[\{x_i\}]$. The methods of proof are the usual ones customary in additive ideal theory. *O. F. G. Schilling* (Chicago, Ill.).

Asano, Keizô. Über Ringe mit Vielfachenkettensatz. Proc. Imp. Acad., Tokyo **15**, 288–291 (1939). [MF 1135]

This paper investigates relations between the chain conditions for ideals in any ring $\mathfrak{o}$. Hopkins' result [Duke Math. J. **4**, 664] that any subring of $\mathfrak{o}$ containing nilpotent elements only is itself nilpotent is proved under the weaker assumptions that the two-sided ideals of $\mathfrak{o}$ and its left ideals modulo each prime ideal satisfy the descending chain conditions. If $\mathfrak{o}$ has non-zero divisors and the descending chain condition for left ideals holds, then the ascending chain condition for left ideals holds. An example is given of a ring in which both chain conditions are satisfied for left ideals but neither holds for right ideals. *N. Jacobson.*

Levi, F. W. Pairs of inverse moduls. J. Indian Math. Soc. **3**, 295–306 (1939). [MF 1446]

The submodules (subgroups under addition) A and A' of a field F are inverse if A contains the inverses of the non-zero elements in A', and vice versa. If the characteristic of F is not equal to 2, it is shown that such modules have the form $A=aK$, $A'=a^{-1}K$, where K is a subfield. A is self-inverse if a^2 is in K, otherwise A and A' have no elements other than 0 in common. If F has characteristic 2, the set K in the representation of A need not be a field but merely a self-inverse module containing 1. A determination of these is given. *N. Jacobson* (Chapel Hill, N. C.).

Krasner, Marc. Remarque au sujet d' "Une généralisation de la notion de corps" (Journ. de Math., 1938, p. 367–385). J. Math. Pures Appl. **18**, 417–418 (1939). [MF 1278]

Becker, M. F. and MacLane, S. The minimum number of generators for inseparable algebraic extensions. Bull. Amer. Math. Soc. **46**, 182–186 (1940). [MF 1268]

If K is a field of characteristic $p>0$ and the order $(K:K^p)=p^m$, then m is the degree of imperfection of K. If L is an algebraic extension of K and e is the least integer such that for each a in L, a^{p^e} is separable over K, then e is the exponent of L over K. It is shown that, if m is finite, every finite algebraic extension L may be generated over K by not more than m elements and there exist extensions that require this number. Any algebraic extension has the same or a smaller degree of imperfection according as it has finite or infinite exponent over K. If L is a purely inseparable finite extension of K, the minimum number of generators of L over K is r determined by $(L:L^p(K))=p^r$. *N. Jacobson* (Chapel Hill, N. C.).

Jacobson, N. The fundamental theorem of the Galois theory for quasi-fields. Ann. of Math. **41**, 1–7 (1940). [MF 1002]

Let P be a quasi-field (that is, non-commutative field or division-ring), and $\mathfrak{G}$ a group of n outer automorphisms $1, S, T, \cdots$ of P. The elements of P which are invariant under all transformations of $\mathfrak{G}$ form a sub-quasi-field Φ of P. The author extends the fundamental theorem of Galois theory to this case. There is a (1-1) correspondence between subgroups $\mathfrak{H}$ of $\mathfrak{G}$ and sub-quasi-fields Σ, $\Phi\leqq\Sigma\leqq P$, such that Σ consists of the elements of P invariant under $\mathfrak{H}$, and $\mathfrak{H}$ is the subgroup of $\mathfrak{G}$ leaving every element of Σ invariant. We may form the "crossed product" $(P, \mathfrak{G})$ consisting of the elements $\sum S\xi_S=\mathbf{W}$ (S in $\mathfrak{G}$, ξ elements of a ring $\mathbf{P}$ isomorphic to P); we have $\xi S=S\xi^S$, where ξ^S is the transform of ξ under S. These $\mathbf{W}$ can be interpreted as endomorphisms of P taken as additive group: $\alpha\rightarrow\alpha\mathbf{W}=\sum\alpha^S\xi_S$. Then $(P, \mathfrak{G})$ is a complete matric ring of degree n over a quasi-field $\mathbf{\Phi}$ isomorphic to Φ. The given quasi-field P forms an n-dimensional vector space over $\mathbf{\Phi}$. More generally, $(P:\mathbf{\Sigma})=(\mathfrak{H}:\{1\})$ and $(\Sigma:\mathbf{\Phi})=(\mathfrak{G}:\mathfrak{H})$ as in ordinary Galois theory. Let Γ be the center of P. The group $\mathfrak{G}$ induces a group $\mathfrak{G}'$ of transformations of Γ which leave fixed the elements of $\Delta=\Gamma\cap\Phi$. If $(P:\Gamma)$ is finite, then $\mathfrak{G}'$ is isomorphic to $\mathfrak{G}$; in the case of infinite rank elements not equal to 1 of $\mathfrak{G}$ may induce the identical mapping in $\mathfrak{G}'$. If $\mathfrak{G}$ and $\mathfrak{G}'$ are isomorphic, then Γ is separable and normal of degree n over Δ, the center of Φ is Δ, and we have $P=\Phi\times\Gamma$ (over Δ). Conversely, if P has this structure, then the automorphisms of the Galois group $\mathfrak{G}'$ of Γ over Δ determine a group $\mathfrak{G}$ of n outer automorphisms of P over Φ. Finally, a theorem of A. Speiser [Math. Z. **5**, 1–6 (1919)] is extended to the case of quasi-fields. *R. Brauer* (Toronto, Ont.).

THEORY OF NUMBERS

Norton, H. W. The 7 x 7 squares. Ann. Eugenics 9, 269–307 (1939). [MF 1467]

An "$n\times n$ Latin square" is a square arrangement of n^2 letters, n of each of n kinds, such that each letter occurs once in each row and once in each column. The number of such squares was found for $n<6$ by Euler, and for $n=6$ by R. A. Fisher and F. Yates [Proc. Cambridge Philos. Soc. 30, 492-507 (1934)]. Two Latin squares are said to be of the same species if one can be derived from the other by permuting rows, columns or letters. Let ij denote the letter that occurs at the intersection of row i and column j in a given Latin square. If it happens that the letters ij and kl are the same, and likewise the letters il and kj, we can obtain another Latin square by interchanging these two letters (in those four places alone). Two species are said to belong to one family if a typical square of one can be derived from a typical square of the other by such an interchange. The author's eleven-page enumeration of 7×7 Latin squares exhibits 146 species, of which 144 belong to one family, while the other two form families by themselves. One of these two is the family of cyclic squares, the most obvious kind of Latin square. The three families comprise 61,428,210,278,400 squares; it is conjectured that no further families exist. By judicious pairing of Latin squares (belonging to six of the 146 species) the author obtains seven species of Graeco-Latin (or Eulerian) squares, representing a totality of 6,263,668,776,960,000 Graeco-Latin squares.

H. S. M. Coxeter (Toronto, Ont.).

Bose, R. C. On the construction of balanced incomplete block designs. Ann. Eugenics 9, 353–399 (1939). [MF 1460]

This is the combinatorial problem of arranging v objects in b sets of k (distinct) objects, so that every object occurs in r of the sets, while every two objects occur together in λ of the sets. Clearly $bk=vr$, $\lambda(v-1)=r(k-1)$. Such arrangement (T_1), with $v=6t+3$, $b=(2t+1)(3t+1)$, $k=3$, $r=3t+1$, $\lambda=1$, is provided by any solution of Kirkman's problem of the $6t+3$ schoolgirls [Rouse Ball, Math. Recreations and Essays **1939**, 267–298]. Another (S_1), with $v=b=4n-1$, $k=r=2n-1$, $\lambda=n-1$, is provided by any "anallagmatic tessellation" [R. E. A. C. Paley, J. Math. Phys. Mass. Inst. Tech. 12, 311–320 (1933)]. The finite Euclidean geometry $EG(N, p^n)$ gives an arrangement with $v=p^{Nn}$, $k=p^{mn}$; and the finite projective geometry $PG(N, p^n)$ gives one with $v=1+p^n+p^{2n}+\cdots+p^{Nn}$, $k=1+p^n+p^{2n}+\cdots+p^{mn}$. Many other series of arrangements (several of which are quite new) are derived by the methods of "symmetrically repeated differences" and "block section."

H. S. M. Coxeter (Toronto, Ont.).

Cox, Gertrude M. Enumeration and construction of balanced incomplete block configurations. Ann. Math. Statistics 11, 72–85 (1940). [MF 1612]

This is a summary of the present state of knowledge on "balanced incomplete block designs" [cf. the foregoing review]. A table is given of the fifty designs most suitable for practical application, together with eleven "arithmetically possible" designs which have never been constructed. There are many simple examples to illustrate the various methods of construction; for example, an Eulerian square of the third order is used to construct a design equivalent to the configuration of inflections of a cubic curve in the complex projective plane, and it is pointed out that the same design can be derived from the finite geometry $EG(2, 3)$. The paper ends with a comprehensive bibliography.

H. S. M. Coxeter (Toronto, Ont.).

Sprague, R. Zur Theorie der Umfüll-Aufgaben. Jber. Deutsch. Math. Verein. Abt. 2. **49**, 65–73 (1940). [MF 1219]

In the theory of mathematical games and puzzles the following problem has been often treated: Given three vessels of content a, b, c, respectively, a, b, c positive integers, a even, $a>b>c$. The vessel a is originally full, b, c empty. The contents are to be divided into two equal parts $a/2$ by continued pouring from one vessel into another. The author starts from a known result: Sufficient for solvability is $b+c-2\leqq a\leqq 2b+2c$; $a\leqq 2b+2c$ is also necessary; but examples show that $b+c-2\leqq a$ is not necessary. This gap is closed in the paper by the final theorem: Assume $|pb-qc|=1$ for integers $p\geqq0$, $q\geqq0$. The problem is solvable when and only when the fraction $(p+q)/(a+b)$, if inserted in its proper place in order of magnitude in the Farey series belonging to $b+c-a=k$, stands next to a (reduced) fraction g/k with $g\equiv p+q$ (mod 2).

A. J. Kempner.

Thébault, V. Curiosités arithmétiques. Mathesis 54, 5–8 (1940). [MF 1574]

Problems and examples of squares and cubes with curious digital properties.

D. H. Lehmer (Bethlehem, Pa.).

Bush, L. E. An asymptotic formula for the average sum of the digits of integers. Amer. Math. Monthly 47, 154–156 (1940). [MF 1735]

Let $S(N)$ be the sum of the digits of all positive integers less than N when these numbers are written in the scale of notation of base r. The author proves that $S(N)/N \sim (r-1)\log N/(2\log r)$. He concludes that, for sufficiently large N, $S(N)/N$ is least in the binary scale.

H. W. Brinkmann (Swarthmore, Pa.).

Pillai, S. S. On m consecutive integers. I. Proc. Indian Acad. Sci., Sect. A. 11, 6–12 (1940). [MF 1498]

The author verifies the following statements: In every set of less than 17 consecutive integers there is a number which is prime to all the others. There are sets of m consecutive integers ($m=17, 18, 19, \cdots, 430$) that have not this quality. The elementary proofs make use of the numerical qualities of the small primes.

P. Scherk.

Marshall, J. B. On the extension of Fermat's theorem to matrices of order n. Proc. Edinburgh Math. Soc. (2) 6, 85–91 (1939). [MF 1520]

The author proves that, if p is a prime number and A is a matrix of order n with integral elements such that A is prime to p, then $A^q\equiv1$ (mod p), where $q=p^r q_n$, p^r being the lowest power of p which is greater than or equal to n, and q_n being determined by the recurrence relation $q_1=p-1$, $q_n=$L.C.M. of q_{n-1} and p^n-1. It is not shown, except in special cases, that q so defined is the smallest possible.

N. H. McCoy (Northampton, Mass.).

Vandiver, H. S. On general methods for obtaining congruences involving Bernoulli numbers. Bull. Amer. Math. Soc. 46, 121–123 (1940). [MF 1258]

If

$$\sum_{i=1}^{s} a_i C_{x_i, i} \equiv 0 \pmod{p^{k-i}} \quad \text{for } i=0, 1, \cdots, k-1,$$

$C_{x,a}=0$ if $a>x$, then

$$\sum_{i=1}^{s} \frac{a_i b_{n+(p-1)x_i}}{n+(p-1)x_i} \equiv 0 \pmod{p^k, p^{n-1}};$$

b_m are the Bernoulli numbers defined by the symbolic formula $(b+1)^n=b^n$. The author indicates a proof and also a new proof of a congruence by Beeger [Bull. Amer. Math. Soc. 44, 684–686 (1938)]. He makes interesting remarks about repetitive sets (mod m), that is, sets $n_1, n_2, \cdots, n_s$ for which a multiplier n exists such that $nn_1, nn_2, \cdots, nn_s$ are congruent (mod m) to $n_1, n_2, \cdots, n_s$ in some order [cf. Vandiver, Ann. of Math. (2) 18, 106 (1917)].

N. G. W. H. Beeger (Amsterdam).

Tchacaloff, Lhristo et Karanicoloff, Chr. Résolution de l'équation $Ax^m+By^n=z^p$ en nombres rationnels. C. R. Acad. Sci. Paris 210, 281–283 (1940). [MF 1597]

Let A and B be given rational numbers, and let m, n, p be integers, relatively prime in pairs. The author considers the problem of finding all rational solutions of the equation $Ax^m+By^n=z^p$. He shows that all rational solutions of this equation may be deduced from the rational solutions of the simple equation $AX+BY=Z$. Here x, y, z and X, Y, Z are connected by a functional relation which depends on m, n and p.

A. C. Schaeffer (Palo Alto, Calif.).

James, R. D. Integers which are not represented by certain ternary quadratic forms. Duke Math. J. 5, 948–962 (1939). [MF 830]

This paper contains the proof of the following theorem: Let any negative integer d be written as $-2^b S^2 h$, where S and h are odd, and h is quadratfrei. Then every sufficiently large integer N can be represented in the form

$$N=u^2+epH^2M,$$

where u is a positive integer, p is a prime such that $(d|p)=-1$, every prime dividing H also divides d, every prime q dividing M satisfies $(d|q)=1$, and e is 1 except in the three following cases: (1) b odd, $h|N$, $h\equiv 3 \pmod 4$, $N\equiv 6 \pmod 8$; (2) b even, $h|N$, $h\equiv 1 \pmod 4$, $N\equiv 2 \pmod 4$; (3) b odd, $h|N$, $h\equiv 1 \pmod 4$, $N\equiv 2 \pmod 8$, in all of which $e=2$.

The proof is accomplished by an application of the Viggo Brun method, which the author quotes in Estermann's presentation [J. Reine Angew. Math. 168, 106–116 (1932)]. The multiplicative function which regulates the sieving in this case is

$$w(q)=\prod_{\substack{p|q \\ (d|p)=-1}} \left\{\frac{1+(D|p)}{p}\right\},$$

where $(d|p)$ is the Kronecker symbol. The author gives in his Lemma 5 a slight improvement of the numerical constants included in Estermann's principal lemma. The crucial step of the proof is about as follows: It has been shown by Viggo Brun's sieve method that to every sufficiently large integer N there exists an integer $u<N^{\frac{1}{2}}$ of the following properties: (1) setting $N-u^2=K\cdot Q$ with $(K, Q)=1$, where K contains only prime factors of d and Q is prime to d, then we have (*) $(d|Q)=-1$; (2) every prime p with $(d|p)=-1$ dividing $N-u^2$ is greater than $N^{\frac{1}{3}}$. Then, because of (2), at most 2 primes p_1 and p_2 with $(d|p_1)=(d|p_2)=-1$ can divide $N-u^2$. But since the product of these primes divides Q, and since (*) holds, only one prime of such a sort can exist. Therefore $Q=p\cdot M$, with $(d|p)=-1$, and where $(d|q)=1$ for all primes q dividing M.

The introductory remarks of the paper, which connect the main theorem with the problem mentioned in the title of the paper, are incomplete and not conclusive. (The author, in a letter to the reviewer, explains that those remarks refer only to sufficiently large integers "of a certain form," as, for example, in the case $d=-3$ to the integers of the form $9^k(9l+6)$.)

H. Rademacher (Swarthmore, Pa.).

Vandiver, H. S. Note on Euler number criteria for the first case of Fermat's last theorem. Amer. J. Math. 62, 79–82 (1940). [MF 958]

Take $m=4$ in a formula of the writer [Ann. of Math. 26, 88–94 (1924)], note that

$$-w\frac{w^{4l}-1}{(w-1)^2} \equiv w+2w^2+\cdots+(4l-1)w^{4l-1}=F_2(w) \pmod l,$$

and let $w=t:\rho^{(w-1)^2}$, $\rho^4=1$, $\rho\neq 1$, then

$$(1) \qquad \sum_\rho \rho(t^2+(\rho+1)t+\rho)^2\cdot\frac{f_{l-2}(\rho)}{\rho^l-1} \equiv 0 \pmod l.$$

Now, if

$$(2) \qquad x^l+y^l+z^l=0, \qquad t\equiv x:y \pmod l,$$

the six numbers t, $1/t$, $t-1$, $1/(t-1)$, $(t-1)/t$, $t/(t-1)$ are roots of the congruence (1) of 4th degree. Hence $t^2-t+1\equiv 0$ [which is inconsistent with (2); cf. Pollascek, Akad. Wiss. Wien, S.-B. 126, 1–15 (1917)] or $t=-1, \frac{1}{2}, 2$. Let $t=\frac{1}{2}$ and $t=2$ in (1) and take the difference of the results. From this, by formulas of Frobenius [S.-B. Preuss. Akad. Wiss. 1914, 653–681], follows $E_{(l-3)/2}\equiv 0 \pmod l$ if (2) exists.

N. G. W. H. Beeger (Amsterdam).

MacLane, Saunders and Schilling, O. F. G. Normal algebraic number fields. Proc. Nat. Acad. Sci. U. S. A. 26, 122–126 (1940). [MF 1280]

Soit K un corps de nombres algébriques. Si, pour chaque diviseur premier $\mathfrak{p}$ de k, on se donne d'une manière quelconque une classe normale simple $H_\mathfrak{p}$ d'algèbre sur $k_\mathfrak{p}$, on obtient ce qu'on appelle une algèbre idéale H sur k. Soit K/k une extension galoisienne finie de k: on dit qu'elle décompose H si, pour chaque $\mathfrak{p}$, $Kk_\mathfrak{p}$ décompose $H_\mathfrak{p}$. M étant un diviseur entier, on dit que H est première à M si $H_\mathfrak{p}=k_\mathfrak{p}$ pour les $\mathfrak{p}$ qui figurent dans M. M étant convenablement choisi en fonction de K, les auteurs calculent l'indice dans le groupe des algèbres idéales décomposées par K et premières à M du sous-groupe composé des classes d'algèbres simples sur k possédant les mêmes propriétés. Cet indice est égal au p.p.c.m. des ordres des éléments du groupe de Galois Γ de K/k. Les auteurs donnent également des formules reliant cet indice aux nombres de classes dans K et k, et aux systèmes de facteurs de Γ dans K. Ces formules généralisent la "première inégalité fondamentale" de la théorie du corps de classes. Enfin, les auteurs annoncent que leurs résultats permettent de définir un symbole de Artin, et un symbole de restes normiques pour les extensions galoisiennes quelconques. Les démonstrations des résultats énoncés paraîtront ultérieurement.

C. Chevalley.

★**Gupta, Hansraj. Tables of partitions.** Indian Mathematical Society, Madras, 1939. viii+81 pp.

These tables are in two main parts, the first [pp. 13–20] giving the number $p(n)$ of unrestricted partitions of n for each $n \leqq 600$, the second part [pp. 23–79] giving the number (n, m) of partitions of n in which the smallest summand or part is equal to m, for $n \leqq 300$ and $2 \leqq m \leqq [n/5]$. A one page table at the end gives the number of those partitions of n in which the smallest part exceeds $[n/4]$. For the values of m omitted, that is for $[n/5] < m \leqq [n/4]$, we may use the explicit formula

$$(n, m) = 3 - 2m + [((n-4m)^2 + 6n)/12] + [(n-3m)/2].$$

The introduction contains an interesting account of the recent history of Ramanujan's conjectured divisibility properties of the function $p(n)$ together with a reproduction of the reviewer's table of the coefficients $A_k(n)$ of the Hardy-Ramanujan, Rademacher series for $p(n)$.

D. H. Lehmer (Bethlehem, Pa.).

Dyer-Bennet, John. A theorem on partitions of the set of positive integers. Amer. Math. Monthly **47**, 152–154 (1940). [MF 1734]

This note has nothing to do with partitions in the ordinary sense of the word, but gives the solution of the problem of finding all moduli m such that $a^\alpha \equiv b^\beta \pmod{m}$ whenever $a \equiv b$, $\alpha \equiv \beta \pmod{m}$. It is shown that m is square free and divisible by $p-1$ for every prime factor p of m. All solutions are then seen to be $m = 1, 2, 6, 42$ and 1806.

D. H. Lehmer (Bethlehem, Pa.).

Niven, Ivan. On a certain partition function. Amer. J. Math. **62**, 353–364 (1940). [MF 1770]

Using the fundamental methods introduced by Hardy and Ramanujan and improved by Rademacher, the author determines the coefficients a_m in the expansion

$$F(x) = \frac{f(x)f(x^6)}{f(x^2)f(x^3)} = \sum_{m=0}^{\infty} a_m x^m,$$

where $f(x) = \prod_{j=1}^{\infty}(1-x^j)^{-1}$. The final result expresses a_m as an infinite series involving Bessel functions with purely imaginary arguments. Since the function $F(x)$ is connected with a modular form of dimension zero it is necessary in the course of the proof to make use of estimates for Kloosterman sums. It is clear from their definition that the a_m are the number of partitions of m into summands of the form $6n \pm 1$. This partition function a_m is of particular interest because I. Schur [S.-B. Preuss. Akad. Wiss. **1926**, 448–495] has shown that it also enumerates the number of partitions of m into summands whose differences are at least three, and at least six if both summands are divisible by three.

H. S. Zuckerman (Seattle, Wash.).

Shah, S. M. An inequality for the arithmetical function $g(x)$. J. Indian Math. Soc. **3**, 316–318 (1939). [MF 1576]

Let $f(n)$ be the maximum least common multiple of $a_1, a_2, \cdots, a_p$, for all partitions $n = a_1 + \cdots + a_p$ in positive integers a_i. Landau [Primzahlen, vol. 1, 222–229] had proved for $g(x) = \log f(x)$ that $g(x) \sim (x \log x)^{\frac{1}{2}}$. Let p_r denote the rth prime, and $y = p_{s+1} - 1$, where $p_1 + \cdots + p_s \leqq x < p_1 + \cdots + p_{s+1}$; set $I(y) = \log p_1 + \cdots + \log p_s$; it is known that $I(y) = (x \log x)^{\frac{1}{2}}\{1 + (\log\log x)/(2\log x)\} + O(x/\log x)^{\frac{1}{2}}$. It is proved that $I(y) \leqq g(x) \leqq I(y) + O(x \log x)^{\frac{1}{4}}$. *G. Pall.*

Tricomi, Francesco. Sulla frequenza dei numeri interi decomponibili nella somma di due potenze k-esime. Atti Accad. Sci. Torino **74**, 369–380 (1939). [MF 1317]

The author develops an approximate formula for the number $N_k(x)$ of numbers not greater than x which are sums of two positive kth powers. The method is based on probability considerations, the independence of certain probabilities being assumed. The general result obtained is

$$N_k(x) = x - \int_0^x \exp\left(-A_k t^{(2-k)/k}\right) dt,$$

where

$$A_k = \frac{\Gamma^2(k^{-1})}{2k^2\Gamma(2k^{-1})}.$$

For $k=2$, we obtain $N_2(x) = (1 - e^{-\pi/8})x = .32477x$, whereas it is known that $N_2(x) = .764x/\log x$. For $k=3$, the result is

(1) $N_3(x) = x(1 - e^{-X}) + \frac{1}{2}A_3 x^{\frac{2}{3}}(1-X)e^{-X} - \frac{1}{2}A_3^3 Ei(-X),$

where $X = A_3 x^{-\frac{1}{3}}$. Graphs are given comparing the actual step-functions $N_2(x)$ and $N_3(x)$ with their approximations for $x \leqq 1000$ and $x \leqq 2000$, respectively. At $x = 1000$, $N_2(x) = 307$ is decidedly below its approximation 324.8. For $x \leqq 2000$, the deviation of $N_3(x)$ from its approximation (1) is much less (not exceeding 3) and the agreement is described as "straordinariamente soddisfacente." Incidentally, the reviewer notes that $N_3(39050) = 495$, whereas (1) gives 504.01.

D. H. Lehmer (Bethlehem, Pa.).

Lévy, Paul. Observations sur le mémoire de M. F. Tricomi: "Sulla frequenza dei numeri interi decomponibili nella somma di due potenze k-esime." Atti Accad. Sci. Torino **75**, 177–183 (1939). [MF 1886]

In the paper reviewed above Tricomi has obtained by a simple heuristic argument, based on probability, the approximate result $N_2(x) = (1 - e^{-\pi/8})x$, a result at variance with Landau's asymptotic formula:

(1) $$N_2(x) \sim \frac{bx}{(\log x)^{\frac{1}{2}}},$$

where the constant b is given by the product

$$b = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod 4} (1 - p^{-2})^{-\frac{1}{2}}.$$

In the present note the author derives (1) by a more elaborate probability argument, admittedly heuristic. The constant b is not determined, however. He also suggests that the probability, for a random x, that $N(x) = \nu$ is given by

$$\frac{b\theta^\nu}{e^\theta \nu! (\log x)^{\frac{1}{2}}}, \quad \text{where} \quad \theta = \frac{\pi(\log x)^{\frac{1}{2}}}{8b}.$$

D. H. Lehmer (Bethlehem, Pa.).

Khintchine, A. Sur la sommation des suites d'entiers positifs. Rec. Math. (Moscou) [Mat. Sbornik] N.S. **6** (48), 161–166 (1939). (Russian. French summary) [MF 1440]

Let φ be a sequence of integers and $\varphi(n)$ the number of elements not greater than n of this sequence; the density of φ (Schnirelmann) is the lower bound $D(\varphi)$ of $\varphi(n)/n$. One writes $\psi = \sum_{r=1}^{h} \varphi_r$ if for every integer $n \epsilon \psi$ there exist integers n_r such that $n_r \epsilon \varphi_r$ $(r = 1, 2, \cdots, h)$ and $n = \sum_{r=1}^{h} n_r$. The conjecture that $\sum_{r=1}^{h} D(\varphi_r) \leqq 1$ implies $D(\sum_{r=1}^{h} \varphi_r) \geqq \sum_{r=1}^{h} D(\varphi_r)$

is still unproved. The present author proved it, however, in the particular case when all the $D(\varphi_r)$'s are equal [Zur additiven Zahlentheorie, Rec. Math. (Moscou) [Mat. Sbornik] N.S. **39**, 27–34 (1932)]. In this paper a simpler proof of this theorem is given. *M. Kac* (Ithaca, N. Y.).

Schnirelmann, L. On addition of sequences and sets. Rec. Math. (Moscou) [Mat. Sbornik] N.S. **5** (47), 211–215 (1939). (English. Russian summary) [MF 1431]

The author proves in this posthumous paper the following theorem: Let A and B be two sets of positive integers. Denote the terms of the sequence A by m_i and those of the sequence B by n_j. Let the number of terms of the sequences A and B not exceeding x be M_x and N_x, respectively. Denote further by $D(A)$ the density of the sequence A, that is,

$$D(A)=\min M_x/x, \qquad x=1, 2, \cdots.$$

By $\overline{A+B}$ we denote the sequence of all integers of the form m_i+n_j, m_i+n_j-1. Then

$$D(\overline{A+B})\geqq\min [1, D(A)+D(B)].$$

Denote by $(A+B)$ the sequence whose terms are the integers of the form m_i, n_j, m_i+n_j. Khintchine conjectured that

$$D(A+B)\geqq\min [1, D(A)+D(B)].$$

It is easy to see that this result of Schnirelmann would follow from the conjecture of Khintchine. Schnirelmann proved these results in the years 1932 or 1933. It can be remarked that this result can easily be deduced from a result of Besicovitch [J. London Math. Soc. **10**, 246–248 (1935)]. *P. Erdös* (Princeton, N. J.).

Mahler, Kurt. Ein Übertragungsprinzip für lineare Ungleichungen. Časopis Pěst. Mat. Fys. **68**, 85–92 (1939). [MF 1083]

The author proves the following generalization of Khintchine's "Übertragungssatz" [Rend. Circ. Mat. Palermo **50**, 170–195 (1926)]. Let $f_1, \cdots, f_n$ be linear forms in $x_1, \cdots, x_n$ with real coefficients and determinant $d\neq 0$, and let $g_1, \cdots, g_n$ be the contragredient linear forms in $y_1, \cdots, y_n$, so that $\sum f_i g_i=\sum x_i y_i$. Suppose there exist integral $x_1, \cdots, x_n$, not all zero, for which

$$|f_1|=t_1, \quad |f_2|\leqq t_2, \cdots, \quad |f_n|\leqq t_n.$$

Then there exist integral $y_1, \cdots, y_n$, not all zero, for which

$$|g_1|\leqq(n-1)\lambda/t_1, \quad |g_2|\leqq\lambda/t_2, \cdots, \quad |g_n|\leqq\lambda/t_n,$$

where $\lambda^{n-1}=t_1t_2\cdots t_n/|d|$. The proof is very simple, and uses only Minkowski's classical theorem on linear forms. It follows as a corollary that if $\Pi|f_i|$ can be made arbitrarily small, so also can $\Pi|g_i|$. The author also proves an analogous "Übertragungssatz" for linear forms with p-adic coefficients. *H. Davenport* (Manchester).

Mahler, Kurt. Ein Übertragungsprinzip für konvexe Körper. Časopis Pěst. Mat. Fys. **68**, 93–102 (1939). [MF 1133]

The author generalizes the "Übertragungsprinzip" for linear forms [see preceding review] to relations between an arbitrary convex body K with center at the origin and the polar reciprocal body K'. Let $F(\mathbf{x})$ ($\mathbf{x}$ any point in n-dimensional space) be the distance-function corresponding to K, so that K is defined by $F(\mathbf{x})\leqq 1$. The distance-function $G(\mathbf{x})$ for K' is given by

$$G(\mathbf{x})=\max_{\mathbf{y}\neq 0}\frac{|\mathbf{x}\cdot\mathbf{y}|}{F(\mathbf{y})}, \quad \mathbf{x}\cdot\mathbf{y} \text{ the scalar product.}$$

Let $\sigma_1=F(\mathbf{x}_1), \cdots, \sigma_n=F(\mathbf{x}_n)$ be the successive lattice-point minima associated with K [see Minkowski, Geometrie der Zahlen, 1910, Kap. 5], and $\tau_1=G(\mathbf{y}_1), \cdots, \tau_n=G(\mathbf{y}_n)$ be those associated with K'. The author proves that

$$(1) \qquad \sigma_h\tau_{n-h+1}\geqq 1, \qquad h=1, 2, \cdots, n.$$

The proof depends on the existence of $i\leqq h, j\leqq n-h+1$ with $|\mathbf{x}_i\mathbf{y}_j|\geqq 1$. In the proof of this, at the bottom of p. 99, the phrase following "also auch" should read "jeder Punkt der durch 0, $x^{(1)}, \cdots, x^{(h)}$ gelegten linearen h-dimensionalen Mannigfaltigkeit $L^{(h)}$ auf jedem Punkt der durch. . . ."

The author also proves by elementary geometrical considerations that the volumes J, J' of K, K' satisfy

$$(2) \qquad 4^n/(n!)^2\leqq JJ'\leqq 4^n,$$

and conjectures the more precise inequality

$$4^n/n!\leqq JJ'\leqq\pi^n/\Gamma(n/2+1)^2,$$

the two sides corresponding to the parallelepiped and to the ellipsoid. From (1), (2) and Minkowski's inequality

$$2^n/n!J\leqq\sigma_1\cdots\sigma_n\leqq 2^n/J,$$

it follows that

$$(1') \qquad 1\leqq\sigma_h\tau_{n-h+1}\leqq(n!)^2.$$

H. Davenport (Manchester).

Mahler, Kurt. Note on the sequence $\sqrt{n}$ (mod 1). Nieuw Arch. Wiskde **20**, 176–178 (1940). [MF 1095]

Let ξ be any real irrational number, η any real number, ϵ any positive number, c a suitable absolute constant. Theorem 1: the inequality

$$|\xi-x^{\frac{1}{2}}-y|<\frac{1+\epsilon}{2\sqrt{5}\cdot x}$$

has an infinity of integral solutions x, y with $x>0$. Theorem 1a: the inequality

$$|\xi-(x+\eta)^{\frac{1}{2}}-y|<c/x$$

has an infinity of similar solutions. These are deduced from a theorem of Khintchine [Math. Ann. **111**, 631–637 (1935)]. On the other hand, if ξ is rational, there exists $c=c(\xi)>0$ such that, for all integers x, y ($x>0$), either

$$|\xi-x^{\frac{1}{2}}-y|\geqq c/x^{\frac{1}{2}} \quad \text{or} \quad \xi-x^{\frac{1}{2}}-y=0.$$

H. Davenport (Manchester).

Koksma, J. F. Über die asymptotische Verteilung gewisser Zahlfolgen modulo eins. Nieuw Arch. Wiskde **20**, 179–183 (1940). [MF 1096]

The author proves the following modification of Theorem 1 of the preceding paper. Let ξ be a real irrational number, but not an algebraic integer of degree 2. For any $\epsilon>0$, at least one of the two inequalities

$$|\xi-x^{\frac{1}{2}}-y|<(1+\epsilon)/8x, \quad |\xi+x^{\frac{1}{2}}-y|<(1+\epsilon)/8x$$

has an infinity of integral solutions with $x>0$. Using Vinogradov's work, an inequality is also given for $|\xi-x^{1/q}-y|$ which has an infinity of solutions. *H. Davenport.*

Koksma, J. F. Ueber die asymptotische Verteilung eines beliebigen Systems (f_r) von n reellen Funktionen f_r der m ganzzahligen Veränderlichen $x_1, x_2, \cdots, x_m$ modulo Eins. Nederl. Akad. Wetensch., Proc. **43**, 211–214 (1940). [MF 1593]

The author proves a general theorem on distributions mod 1; his result in the simplest case of one dimension is the following one: Let $f(1), f(2), \cdots$ be an arbitrary sequence of

real numbers, $\varphi(1)$, $\varphi(2)$, $\cdots$ a sequence of positive numbers such that $\sum_{x=1}^{\infty}\varphi(x)$ converges. Then for "nearly all" real α the Diophantine inequalities

$$-\varphi(x)\leqq f(x)-y-\alpha\leqq\varphi(x)$$

have only a finite number of integral solutions $x\geqq 1$, y.

K. Mahler (Manchester).

Scott, W. T. Approximation to real irrationals by certain classes of rational fractions. Bull. Amer. Math. Soc. **46**, 124–129 (1940). [MF 1259]

Using the method developed by Humbert and Ford (depending on geometric properties of elliptic modular transformations) the author considers the inequality

$$(1)\qquad \left|\omega-\frac{p}{q}\right|<\frac{k}{q^2},$$

ω being a real irrational number and $k>0$ a constant. Each irreducible fraction p/q with (1) belongs to one of the three classes $[o/e]$, $[e/o]$ and $[o/o]$, o denoting an odd integer and e an even integer. The author shows: If $k\geqq 1$ there are infinitely many fractions p/q of each of the classes satisfying (1), regardless of the value of ω. If $k<1$ there exist irrational numbers ω, everywhere dense on the real axis, for which (1) is satisfied by only a finite number of fractions of a given one of the three classes. For the method used, cf. Kap. 3 of the reviewer's "Diophantische Approximationen" [Ergebnisse der Mathematik IV, 4, Berlin, 1936].

J. F. Koksma (Amsterdam).

Spencer, D. C. On a Hardy-Littlewood problem of diophantine approximation. Proc. Cambridge Philos. Soc. **35**, 527–547 (1939). [MF 834]

Let ω_1 and ω_2 be two positive numbers whose ratio $\theta=\omega_1/\omega_2$ is irrational and ξ a real number with $0\leqq\xi<\omega_1$. Let further, for real $r\geqq 1$,

$$P_r(x)=-\sum\frac{2\cos(2\pi nx-\frac{1}{2}\pi r)}{(2\pi n)^r}.$$

The author gives a systematic treatment of the sums

$$R_r=R_r(\xi,m)=\sum_{\nu=0}^{m}P_r\left(\frac{\xi+\nu\omega_2}{\omega_1}\right),$$

which, for positive integers r, have been considered (with different methods) by Hardy-Littlewood, Ostrowski, Behnke and Khintchine [references in the reviewer's "Diophantische Approximationen," Berlin, 1936 (Ergebnisse der Mathematik IV, 4)]. The method depends on a formulation of the problem in terms of a contour integral of the function

$$\frac{e^{-(\omega_1+\omega_2)s}}{(1-e^{-\omega_1 s})(1-e^{-\omega_2 s})}\frac{e^{\eta s}}{s^r},\qquad \eta \text{ real},$$

along the lines of the transcendental method developed by Hardy-Littiewood in their second memoir on the lattice points of a right-angled triangle [Abh. Math. Sem. Hansischen Univ. 1, 212–249 (1922)].

O-results: If θ is of type I hc $(0\leqq c<\infty)$, we have $R_r=O(m^{1-r/h})$ if $r<h$; $R_1=O(\log m)$ if $h=1$; $R_r=O(\log\log m)$ if $h=r>1$ and $R_r=O(1)$ if $r>h\geqq 1$. If θ is of type II hc $(h>0, 0\leqq c<\infty)$, we have $R_r=O(m\log^{-r/h}m)$. If $\epsilon>0$, then for almost all points (ω_1,ω_2) in the plane $R_1=O(\log^{1+\epsilon}m)$; $R_r=O(1)$ if $r>1$. All results hold uniformly in ξ. Ω-result: If $r\geqq 1$, θ of type I hc $(0<c\leqq\infty)$, then for an infinity of m $R_r>\text{const.}\cdot m^{1-r/h}$.

J. F. Koksma (Amsterdam).

Kac, M., van Kampen, E. R. and Wintner, Aurel. Ramanujan sums and almost periodic functions. Amer. J. Math. **62**, 107–114 (1940). [MF 961]

Ramanujan has [Trans. Cambridge Philos. Soc. **22**, 259–276 (1918); Collected Papers, 179–199] obtained expressions of a variety of arithmetical functions $f(n)$ in the form of a series $\sum_{r=1}^{\infty}a_r c_r(n)$, where $c_r(n)$ denotes the sum $\sum_m \cos(2\pi(m/r)n)$ extended over all m in the interval $0<m\leqq r$ which are prime to r. In the present paper it is shown that for a certain class of functions $f(n)$ the trigonometrical series in question are almost periodic Fourier series of the functions in the sense of Besicovitch. The main result is that, if $f(n)$ is a strongly multiplicative function (that is, $f(n)=\Pi_{p|n}f(p)$) for which

$$\sum_p\frac{|f(p)-1|}{p}$$

is convergent, then $f(n)$ is almost periodic in the sense of Besicovitch and has the Fourier series $f(n)\sim\sum_{n=1}^{\infty}a_r c_r(n)$, where

$$a_r=\prod_p\left(1+\frac{f(p)-1}{p}\right)\prod_{p|r}\frac{f(p)-1}{f(p)-1+p}$$

or 0 according as r is or is not squarefree.

B. Jessen.

Rankin, R. A. Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions. III. A note on the sum function of the Fourier coefficients of integral modular forms. Proc. Cambridge Philos. Soc. **36**, 150–151 (1940). [MF 1715]

If $H(\tau)=\sum_1^{\infty}a_n e^{2\pi i n\tau/N}$ (absolutely convergent for $\mathfrak{J}\tau>0$) is an integral modular form of dimensions $-\kappa\leqq -2$ and Stufe N, vanishing at all rational cusps of the fundamental region ("cusp-form"), then

$$\sum_{n\leqq x}a_n=O(x^{\frac{1}{2}\kappa-\frac{1}{5}}).$$

To obtain this improvement of Walfisz's estimate $O(x^{\frac{1}{2}\kappa-1/3+\epsilon})$ [Math. Ann. **108**, 75–90 (1933)] the author follows a similar method, but uses, in place of Kloosterman's theorem $a_n=O(n^{\frac{1}{2}\kappa-\frac{1}{4}+\epsilon})$, his own theorem

$$\sum_{n\leqq x}|a_n|^2=\alpha x^{\kappa}+O(x^{\kappa-\frac{3}{5}}),$$

proved in paper II of the series [Proc. Cambridge Philos. Soc. **35**, 357–372 (1939); cf. these Rev. 1, 69–70 (1940). In this abstract the conjecture of Ramanujan referred to on p. 70, lines 23–24, should read $|\tau(n)|\leqq n^{11/2}d(n)=O(n^{11/2+\epsilon})$].

A. E. Ingham (Berkeley, Calif.).

Schoeneberg, Bruno. Über die ζ-Funktion einfacher hyperkomplexer Systeme. Math. Ann. **117**, 85–88 (1939). [MF 1384]

The zeta-function of a maximal order $\mathfrak{O}$ in a simple algebra $\mathfrak{A}$ over an algebraic number field $\mathfrak{Z}$ as central is expressible in terms of the Dedekind zeta-function of $\mathfrak{Z}$ and elementary functions by a formula due to K. Hey. The original proof, based on Artin's theory of the arithmetic of hypercomplex number systems, is here replaced by one based on Hasse's theory.

A. E. Ingham (Berkeley, Calif.).

Siegel, Carl Ludwig. Einführung in die Theorie der Modulfunktionen n-ten Grades. Math. Ann. **116**, 617–657 (1939). [MF 1336]

In this paper the author continues the paper: "Über die analytische Theorie der quadratischen Formen" [Ann. of

Math. 36, 527–606 (1935)]. Let x_i and y_i $(i=1, 2, \cdots, 2n)$ be two systems of $2n$ cogredient variables; he studies the transformations T

$$(1) \qquad x_i'=\sum_k a_{ik}x_k, \quad y_i'=\sum_k a_{ik}y_k, \quad i, k=1, 2, \cdots, 2n,$$

with integer coefficients, which leave invariant the bilinear form

$$(2) \qquad \sum_1^n (x_iy_{n+i}-y_ix_{n+i}).$$

The group Γ of these transformations is the modular group of genus (or degree) n. This group is arrived at by a generalization of the well-known modular group for the elliptic curves to the curves of genus n. Using the calculus of matrices, Siegel finds, with great simplicity, the algebraic properties of the transformations (1). The matrix corresponding to a transformation (1) is denoted by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are square matrices of n rows and columns. If V, W are two other matrices of order n, the author writes

$$V_1=AV+BW, \quad W_1=CV+DW \quad \text{or} \quad \begin{pmatrix} V_1 \\ W_1 \end{pmatrix}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} V \\ W \end{pmatrix},$$
$$Z=VW^{-1}, \qquad Z_1=V_1W_1^{-1},$$

or, with a symbolic notation,

$$(3) \qquad Z_1=(AZ+B)(CZ+D)^{-1}.$$

He also puts $Z=X+iY$, $Z_1=X_1+iY_1$, where X, Y, X_1, Y_1 are real matrices, and proves that, if Z is symmetric and the imaginary part Y is the matrix of a positive quadratic form, Z_1 also possesses the analogous properties. The coefficients of X, Y are considered as Cartesian coordinates of a point in an Euclidean hyperspace S. If Z satisfies the preceding hypothesis, the corresponding point belongs to an open convex region P, the boundary of which consists of a finite number of algebraic surfaces. The transformations (3) transform this region P into itself. The author succeeds in finding inside P a fundamental region F for the group of the transformations (3); he proves that this fundamental region F is connected and that its boundary also consists of a finite number of algebraic surfaces. Siegel calls the functions $\varphi(Z)$ of the coefficients of the matrices Z (which satisfy the preceding conditions) modular forms if they are regular inside P, bounded inside F, and satisfy, for every transformation (3), the functional equation

$$\varphi(Z_1)=|CZ+D|^g\varphi(Z), \qquad g=\text{const.}$$

He studies not only these forms by means of their Fourier development, but also the algebraic equations, which can connect many modular forms; he makes use also of series which can be considered as a generalization of Eisenstein's series (the difference being that in Siegel's problems coefficients and variables are no longer numbers, but matrices). The ratio of two modular forms, corresponding to the same value of g, is called a modular function. Siegel proves that these functions are the functions of an algebraic field, which contains precisely $n(n+1)/2$ independent functions, and that they can all be obtained by using Eisenstein's series; he studies their development into Fourier series, thus obtaining theorems which are important for arithmetical applications.

It goes without saying that the paper is also important for the analytical theory of the algebraic moduli of an algebraic curve. *G. Fubini* (Princeton, N. J.).

ANALYSIS

Vijayaraghavan, T. On a conjecture of Gillis. Quart. J. Math., Oxford Ser. 10, 320 (1939). [MF 1042]

Let C denote the curve $y=f(x)$ with $f(x)$ continuous and periodic. Suppose C meets the straight line $y=mx+c$ in a finite number of points. Then there are constants m' and c' such that $y=m'x+c'$ meets C in one point only.

[The reviewer remarks that the following more general theorem is easily verified: Let $f(x)$ be continuous and bounded; in order that there exist a straight line $y=mx+c$ cutting $z=f(x)$ in exactly one point x_0, it is necessary and sufficient that either both $D^+f(x_0)$ and $D^-f(x_0)$ or both $D_+f(x_0)$ and $D_-f(x_0)$ are finite. Otherwise every line $y=mx+c$ cuts $y=f(x)$ in infinitely many points, or not at all.] *W. Feller* (Providence, R. I.).

Popoviciu, Tiberiu. Notes sur les fonctions convexes d'ordre supérieur (VI). Rev. Math. Union Interbalkan. 2, 31–40 (1939). [MF 594]

A function $f(x)$ defined in a linear closed set E is called non-concave of order n if its $(n+1)$th divided difference is not less than 0 for all values of the $n+2$ arguments in E [see the author's paper in Mathematica, Cluj 8, 1–85 (1934)]. The main purpose of the present paper is to generalize the following elementary characterization of ordinary non-concave functions (of order 1 in the author's terminology): a function $f(x)$ continuous in a closed interval is non-concave if and only if any function of the form $f(x)+\alpha x+\beta$ in any subinterval attains its maximum at one of the end points. Suppose that E is a closed interval (a, b), then one of the results can be formulated as follows: Let $f(x)$ be not constant and non-concave of order n in (a, b) and let $E(M)$ be the set at which $f(x)$ attains its maximum. Then $E(M)$ has the following property A: If n is even, $E(M)$ consists of at most $(n/2)+1$ points or is an interval (c, b), where $c>a$, and, if $E(M)$ contains $(n/2)+1$ points, it contains b. If n is odd, $E(M)$ consists of at most $((n+1)/2)+1$ points, and, if it contains at least $(n+1)/2$ points, it contains a or b. Conversely: Let $f(x)$ be continuous in (a, b), E_1 any subinterval of (a, b) and $P(x)$ a polynomial of degree not greater than n. If the set $E_1(M)$ corresponding to E_1 and to any function of the form $f(x)+P(x)$ has the property A, then $f(x)$ is non-concave of order n in (a, b). Similar results are obtained for the set at which $f(x)$ attains its minimum, and related characterizations of the non-concave functions of order n by means of the Tchebycheff approximating polynomials are given. *W. Fenchel.*

Popoviciu, Tiberiu. Notes sur les fonctions convexes d'ordre supérieur (VII). Acad. Roum. Bull. Sect. Sci. 22, 29–33 (1939). [MF 1700]

Popoviciu, Tiberiu. Note sur les fonctions convexes d'ordre supérieur (VIII). Acad. Roum. Bull. Sect. Sci. 22, 34–41 (1939). [MF 1701]

1. Let $f(x)$ be defined on a linear set E. It is said to be convex of order n if (1) $[x_1, x_2, \cdots, x_{n+2}; f]>0$ for an arbitrary set of $n+2$ distinct points $x_1, \cdots, x_{n+2}$ of E, the left-hand side of (1) being the divided difference of $f(x)$ over

those points. Likewise $f(x)$ is called nonconcave, polynomial, nonconvex or concave of order n if it satisfies (1) with the sign $>$ replaced by $\geqq$, $=$, $\leqq$ or $<$, respectively. Any of these functions are said to be of order n. Let now (2) $x_1<x_2<\cdots<x_m$ $(m>n+2)$ be points of E. The author has shown [Mathematica 8, 1–85 (1934)] that if $f(x)$ is of order n then, for each fixed value of $k=0, 1, \cdots, n$, the sequence of differences

$$(3) \qquad [x_i, x_{i+1}, \cdots, x_{i+n-k+1}; f], \quad i=1, \cdots, m-n+k-1,$$

has at most k variations of sign. The chief result of the paper is the following inverse theorem. If, for an arbitrary finite set (2) of E, it is true that the sequence (3) formed for $f-P$, rather than f, where P is an arbitrary polynomial of degree not greater than n, has at most k variations of sign, then f is of order n in E. This gives a separate result for each value of $k=0, 1, \cdots, n$. The proof rests on the fact that a divided difference of f over any $n-k+2$ among the points (2) is an arithmetic mean of the consecutive differences (3).

2. The concept of a $f(x)$ which is convex, nonconcave, $\cdots$, of order n is localized, $f(x)$ being called locally convex, say, of order n in the set of definition E if it has that property in the old sense in some neighborhood of every point of the closure $\bar{E}$ of E. However, in order to make this definition effective for deriving the result of this note that local convexity, nonconcavity, $\cdots$, of order n implies the corresponding property in the large, a certain restriction in the type of neighborhood to be used in the definition of local convexity is needed. For its precise description we refer to the paper. *I. J. Schoenberg* (Waterville, Me.).

Duffin, R. J. and Schaeffer, A. C. On the extension of a functional inequality of S. Bernstein to non-analytic functions. Bull. Amer. Math. Soc. **46**, 356–363 (1940). [MF 1848]

S. Bernstein's theorem referred to in the title is as follows. I. If $\varphi(x)$ is a trigonometric polynomial $\varphi(x)=\sum_0^N(a_\nu \cos \nu x+b_\nu \sin \nu x)$ and if $|\varphi(x)|\leqq 1$ for all x (real), then $|\varphi'(x)|<N$ for all x, unless $\varphi(x)\equiv\sin(Nx+\alpha)$ for some constant α. The authors now prove with strictly elementary means the following general result. II. If a function $f(x)$ satisfies the conditions (1) $(f(x))^2\leqq 1$ and (2) $(f^{(n)}(x))^2+(f^{(n-1)}(x))^2\leqq 1$ for all x and for some positive integer n, then the inequality (2) is valid also when n is replaced by any smaller positive integer. We quote from the paper the way in which II implies I: "Let $f(x)=\varphi(x/\lambda)$, where λ is some constant greater than N. Then as $n\to\infty$, $f^{(n)}(x)\to 0$ uniformly in $(-\infty, \infty)$; hence for sufficiently large n, $(f^{(n)}(x))^2+(f^{(n-1)}(x))^2\leqq 1$. Thus by Theorem II we have $(f^{(k)}(x))^2+(f^{(k-1)}(x))^2\leqq 1$ or

$$(\lambda^{-k}\varphi^{(k)}(x))^2+(\lambda^{-k+1}\varphi^{(k-1)}(x))^2\leqq 1, \quad k=1, 2, 3, \cdots.$$

Considering any fixed k and letting $\lambda\to N+0$, we obtain

$$(N^{-k}\varphi^{(k)}(x))^2+(N^{-k+1}\varphi^{(k-1)}(x))^2\leqq 1, \quad k=1, 2, 3, \cdots."$$

Thus $|\varphi^{(k)}(x)|\leqq N^k$ for all x. The equality sign for some x would imply $\varphi(x)\equiv\sin(Nx+\alpha)$ on account of a general result supplementing Theorem II to the effect that the equality sign in $(f^{(k)}(x))^2+(f^{(k-1)}(x))^2\leqq 1$ for some x and some $k=1, 2, \cdots, n-1$, implies that $f(x)\equiv\sin(x+\gamma)$
I. J. Schoenberg (Waterville, Me.).

Watson, G. N. Three triple integrals. Quart. J. Math., Oxford Ser. **10**, 266–276 (1939). [MF 1037]

The triple integral $\iiint f(u, v, w)dudvdw$, $0\leqq u, v, w\leqq\pi$, is calculated in the following cases: $\{f(u, v, w)\}^{-1}=1-\cos u \cos v \cos w$, $3-\cos v \cos w-\cos w \cos u-\cos u \cos v$, $3-\cos u-\cos v-\cos w$. The corresponding values are $4\pi K_0^2$, $\pi\cdot 3^{\frac{1}{2}}K_1^2$, $4\pi(18+12\cdot 2^{\frac{1}{2}}-10\cdot 3^{\frac{1}{2}}-7\cdot 6^{\frac{1}{2}})K_2^2$, where K_0, K_1, K_2 denote the complete elliptic integrals with moduli $\sin 45°$, $\sin 15°$, $(2-3^{\frac{1}{2}})(3^{\frac{1}{2}}-2^{\frac{1}{2}})$, respectively. In the two first cases a representation in terms of Γ-functions is also possible, which can be used for numerical purposes. In the third case the numerical calculation is possible by means of the representation

$$2\pi^{-1}K_2=\{1+2\sum_{n=1}^{\infty}e^{-6^{\frac{1}{2}}\pi n^2}\}^2.$$

The problem arose in an investigation in ferromagnetism.
G. Szegö (Stanford University, Calif.).

Alessi, Juan M. On Heine's transformation of dual and bidual variables. An. Soc. Ci. Argentina **128**, 222–232 (1939). (Spanish) [MF 1077]

The author considers the transformation

$$(1) \qquad f(z)=\int_0^1\frac{\varphi(t)dt}{t-z},$$

when $z=x+ky$ is a "dual" variable $(k^2=0)$. He separates (1) into its real and "imaginary" parts, which reduces (1) to a sum of two integrals of the classical type, and so is able to extend to (1) the first properties of the classical transformation (holomorphism, sum of transformations, expression of the derivatives, etc.). Similar considerations for the case of a "bidual" variable $z=(x_1+ix_2)+k(x_3+ix_4)$.
A. González Domínguez (Providence, R. I.).

Fubini, Guido. On Cauchy's integral theorem and on the law of the mean for non-derivable functions. Proc. Nat. Acad. Sci. U. S. A. **26**, 199–204 (1940). [MF 1604]

Recently Menger obtained a necessary and sufficient condition in order that the integral $\int pdx+qdy$, where p and q are continuous in a rectangle R, have the same value for any two co-terminal (rectifiable) curves in R [Proc. Nat. Acad. Sci. U. S. A. **25**, 621–625 (1939); these Rev. **1**, 72 (1940)]. In the present note the author derives a new simple necessary and sufficient condition. The condition is again in terms of the existence of nets covering R and having certain properties. Implications on the differentiability properties of p and q are also discussed. The author concludes with some related remarks on the Law of the Mean and on the Fourier coefficients of the developments of p and q.
W. T. Martin (Cambridge, Mass.).

Theory of Sets, Theory of Functions of Real Variables

✶Carathéodory, Constantin. Reelle Funktionen. Band I. Zahlen, Punktmengen, Funktionen. B. G. Teubner, Leipzig, 1939. vi+184 pp. RM 8.40.

The present edition of Carathéodory's classical book differs considerably from the previous second edition. Among the most important changes should be mentioned a new application of the Lindelöf and Borel covering theorems to a general theory of connectedness, and also a new proof [due to R. Rado] of a theorem concerning the extension of the domain of definition of a continuous (or a semi-continuous) function. The theory of functions of bounded

variation is postponed to a next volume. It is announced that the second volume will treat the theory of measure and integration, and the third volume the theory of differentiation and some selected topics of applications of the theory of real functions. The list of contents follows: Chapter I: The real numbers; Chapter II: The notion of limit; Chapter III: Point-sets in the Euclidean space; Chapter IV: The normal covering sequences and the theory of connectedness; Chapter V: Functions; Chapter VI: The distance function and its application; Chapter VII: Sequences of functions.

J. D. Tamarkin (Providence, R. I.).

Sierpinski, Waclaw. Remarque sur les ensembles des nombres ordinaux de classes I et II. Revista Ci., Lima **41**, 289–296 (1939). [MF 1638]

An uncountable sub-class of the class Z_2 of all ordinals less than Ω is said to be rarefied if, when arranged in natural order as $\{\phi_\xi\}_{\xi<\Omega}$, we have $\lim \rho_\xi=\Omega$, where ρ_ξ is defined by $\phi_{\xi+1}=\phi_\xi+\rho_\xi$. A similar definition is given for rarefied subclasses of Z_1, the class of all positive integers. The following results are established: (1) Z_1 is not the sum of a finite set of rarefied subclasses; (2) Z_2 is the sum of a countably infinite set of rarefied subclasses. The decomposition obtained in (2) is not effective; indeed, if it were, the following problem could be solved: Define effectively a law which associates with each $\alpha\epsilon Z_2$ an effectively defined sequence containing all the ordinals less than α. This problem is, in the present state of knowledge, extremely difficult.

J. Todd (Belfast).

Shmushkovitch, V. On a combinatorial theorem of the theory of sets. Rec. Math. (Moscou) [Mat. Sbornik] N.S. 6 (48), 139–147 (1939). (Russian. English summary) [MF 1438]

The following theorem is proved. Let an arbitrary set M be divided in two different manners into an arbitrary number (of any power) of non-empty and mutually disjoint subsets: $M=\sum A_\alpha=\sum A_\beta'$. Besides, let the following conditions be satisfied: (a) p (finite) arbitrary subsets of either subdivision covers completely not more than p subsets of the other; (b) every subset of each subdivision is completely covered by a finite number of subsets of the other subdivision. Then the subdivisions have a common system of representatives. A set $R\subset M$ is called a system of representatives of the subdivision $M=\sum A_\alpha$ if every intersection $R\cdot A_\alpha$ contains one and only one point. There is also a generalization with an extension of the notion of representative set. *J. V. Wehausen* (New York, N. Y.).

Maximoff, Isaiah. On a continuum of power $2^{\aleph_\iota}$. Ann. of Math. **41**, 321–327 (1940). [MF 1815]

The author considers the set of formal continued fractions represented by sequences of type ω_ι consisting of ordinal numbers less than ω_ι, and ordered in the natural way. He proves directly that this set is a natural generalization of the set of real numbers in that (i) it is of power $2^{\aleph_\iota}$ and contains a dense subset of power $\aleph_\iota$, (ii) every Dedekind cut is represented by an element of the set.

J. W. Tukey (Princeton, N. J.).

Sierpinski, W. Sur un théorème de la théorie de la mesure. Proc. Benares Math. Soc. **1**, 35–37 (1939). [MF 1529]

The author proves the following known theorem: there exists an additive measure assuming the values 0 and 1 but no others, defined for all subsets of a fixed infinite set K, and vanishing for all subsets of cardinal number less than that of K. This theorem is equivalent to the theorem: in the Boolean algebra of all subsets of K, the ideal consisting of all subsets of cardinal number less than that of K has a prime ideal divisor. *M. H. Stone* (Cambridge, Mass.).

Price, G. Baley. Definitions and properties of monotone functions. Bull. Amer. Math. Soc. **46**, 77–80 (1940). [MF 1249]

A function $x(t)$ is monotone, by definition, if and only if $x(t)$ is between $x(t_1)$ and $x(t_2)$ whenever t is between t_1 and t_2. Here t is a real variable, $a\leqq t\leqq b$, and betweenness for t is defined in the usual way. The paper is a study of the concepts of monotoneity, variation, etc., according to various definitions of betweenness in the domain X of $x(t)$. The domains X considered are (a) linear partially ordered spaces and partially ordered topological groups, (b) complete metric spaces, (c) metric spaces. The results are linked with the work of Bochner, Graves, Kantorovitch, Menger.

H. E. Bray (Houston, Tex.).

Good, I. J. The approximate local monotony of measurable functions. Proc. Cambridge Philos. Soc. **36**, 9–13 (1940). [MF 894]

The author [following Khintchine, Rec. Math. (Moscou) [Mat. Sbornik] N.S. **31**, 265–285 (1924)] describes a function $f(x)$ as approximately constant at $x=x_0$ if the set in which $f(x)=f(x_0)$ has a point of metric density at x_0, and as approximately increasing (decreasing) in a weak sense if the set in which $\{f(x)-f(x_0)\}/(x-x_0)>(<)0$ has upper density one at x_0. The following theorem is proved: If the measurable function $f(x)$ is defined in a set E, then, at almost all points of E, $f(x)$ is either approximately constant or (in the weak sense) approximately increasing or decreasing. It is deduced that the set of "strict approximate maxima" of a measurable function is of measure zero.

U. S. Haslam-Jones.

Tolstoff, G. Sur quelques propriétés des fonctions approximativement continues. Rec. Math. (Moscou) [Mat. Sbornik] N.S. **5** (47), 637–645 (1939). (French. Russian summary) [MF 1350]

The principal result of this paper is: Let $f(x)$ be continuous on (a, b) and possess an approximate derivative ADf at all points except possibly a denumerable set. If $ADf\geqq 0$ almost everywhere, then $f(x)$ is non-decreasing, and ADf is the ordinary derivative, where it exists, finite or infinite. To demonstrate this the author makes use of five lemmas concerning functions which are approximately continuous and which possess an approximate derivative in all points except perhaps a denumerable set, three of which we state. (1) If $f(x)$ is continuous in all points of a perfect set P with respect to this set, then the sets Q: $AD_+f>p$, R: $AD^+f<q$, $p>q$, cannot be simultaneously non-dense on P. (2) If in all points of a perfect set P, except perhaps a denumerable set, ADf exists and is not $-\infty$, there exists a portion π of P such that if x' and x'' are any two points of π then $\{f(x')-f(x'')\}/(x'-x'')>-k$, where k is a certain constant. (3) If $g(x)$ is approximately continuous and of bounded variation, it is continuous.

Another result is: If $f(x)$ is approximately continuous, except for a denumerable set, and $ADf=0$ almost everywhere, then $f(x)$ is a constant. In conclusion, the paper gives a negative answer to a question proposed by Lusin: If $f(x)$

is an arbitrary measurable function, does there exist a continuous function $F(x)$ with $F'(x)=f(x)$ in a set of the second category with measure equal to $b-a$? *R. L. Jeffery.*

Shukla, Parmeshwar Din. On a non-differentiable function of Denjoy. Proc. Benares Math. Soc. 1, 97–102 (1939). [MF 1536]

Elementary proofs of some properties of a known example. *J. A. Clarkson* (Philadelphia, Pa.).

Popoff, Kyrille. Su una generalizzazione della nozione di derivata di una funzione di variabile reale o complessa. Rend. Sem. Mat. Roma (4) 3, 162–170 (1939). [MF 1703]

The author gives a summary of certain of his previous papers on the same subject [Monatsh. Math. Phys. **48**, 103–120 (1939); C. R. Acad. Sci. Paris **209**, 472–474 and 668–670 (1939); these Rev. **1**, 109 and 115 (1940)].

W. T. Martin (Cambridge, Mass.).

Popoff, Kyrille. Sur une extension de la notion de dérivée. Ann. Univ. Sofia. II. Fac. Phys. Math. Livre 1. **35**, 225–249 (1939). (Bulgarian. French summary) [MF 1376]

This paper is essentially a translation of the paper of the same title published in Monatsh. Math. Phys. **48**, 103–120 (1939) [these Rev. **1**, 109–110 (1940)].

J. D. Tamarkin (Providence, R. I.).

✶Kempisty, Stefan. Fonctions d'intervalle non additives. Actual. Sci. Ind. 824. Ensembles et fonctions. III. Hermann & Cie, Paris, 1939. 62 pp.

The integral $\int_J{}^* f(I)$ of a non-additive function of intervals $f(I)$ over an interval J was defined by Burkill as the limit (in the usual sense) of $\sum f(I_n)$, where $I_1, I_2, \cdots$ are non-overlapping intervals such that $\sum I_n = J$. A similar function was defined to be the "variation" of $f(I)$ by Banach. In this monograph the ideas of Burkill and Banach are further developed and applied to the discussion of areas and lengths, and of the totalization (Denjoy integration) of functions of several variables. For this purpose the author restricts himself mainly to those ("semi-regular") systems of intervals to which the covering theorem of Vitali may be applied: a system of intervals is said to be semi-regular if the ratio $|I|/|K|$ (where K is the smallest "cube" which contains I) is bounded. Thus a restricted Burkill integral $\int_J f(I)$ is defined in which the intervals of subdivision are semi-regular.

In Chapters I–V the general theory is developed; applications are discussed in Chapters VI and VII. Chapter I contains the covering theorems of Vitali and Burkill; II, the Burkill integral and its extensions; III, the differentiation of $f(I)$ in the strict sense (D^*f) and in the semi-regular sense (Df); IV, functions $f(I)$ of bounded variation (VB^* and VB); V, absolutely continuous functions (AC^* and AC). In Chapter VI the author obtains the theorems of Tonelli on rectifiable curves and of de Geöcze and Tonelli on the areas of curved surfaces, and also discusses the area as the limit of the areas of inscribed polyhedra. Finally, in Chapter VII (Fonctions résolubles), he defines both descriptively and constructively a semi-regular $\mathfrak{D}$-integral in k dimensions, which is identical with the Denjoy integral when $k=1$.

U. S. Haslam-Jones (Oxford).

Marcinkiewicz, J. et Salem, R. Sur les sommes riemanniennes. Compositio Math. 7, 376–389 (1940). [MF 1871]

Let $f(x)$ be a function of period 1 and L-integrable over the interval (0, 1). Jessen proved [Ann. of Math. **35**, 248–251 (1934)] that, if $\{n_k\}$ is any sequence of positive integers such that n_{k+1} is divisible by n_k, then

$$F_{n_k}(x)\to\int_0^1 f(t)dt$$

for almost every x, where

$$F_n(x)=\frac{1}{n}\sum_{\nu=0}^{n-1} f\left(x+\frac{\nu}{n}\right).$$

On the other hand, it is known [see Ursell, J. London Math. Soc. **12**, 229–232 (1937), or Marcinkiewicz and Zygmund, Fund. Math. **28**, 131–166 (1937), especially p. 157] that there exist integrable functions f such that

$$\limsup_{n\to\infty} |F_n(x)| = +\infty$$

for every value of x. In the present paper the authors prove a number of results connected with the Jessen theorem. For example:

(a) If

$$\int_0^\pi |f(x+h)-f(x)|^2dx = O(|h|^\epsilon), \qquad \epsilon>0,$$

then

$$F_n(x)\to\int_0^1 f(t)dt$$

for almost every value of x.

(b) If

$$\int_0^1\int_0^1 \frac{|f(x+t)-f(x)|^2}{t|\log t/2|}dtdx<\infty:$$

this is true if, for example,

$$\int_0^1 |f(x+t)-f(x)|^2dx = O\left(\frac{1}{\log^2|\log t|}\right);$$

then

$$(*)\qquad \frac{1}{n}\sum_{n=1}^{n} F_n(x)\to\int_0^1 f(t)dt$$

for almost every x.

(c) If

$$\int_0^1 |f(x+t)-f(x)|dx = O\left(\frac{1}{|\log t|^s}\right), \qquad s>1,$$

then (*) holds. *A. Zygmund* (Cambridge, Mass.).

Denjoy, Arnaud. Totalisation des séries. C. R. Acad. Sci. Paris **209**, 825–828 (1939). [MF 860]

On the interval σ $(0<\theta<1)$ let e be a denumerable set of points θ_n $(n\geqq 1)$, u_n a number placed at θ_n. The problem of totalizing the series $\sum Tu_n$ is that of adding the numbers u_n in the order of increasing magnitude of the corresponding points of the set e. If θ_n increases with n and tends towards unity, then the order of u_n is that of their subscripts, and the total of the series $\sum Tu$ is the sum of the ordinary series $u_1+u_2+\cdots$. In the general case the series $\sum Tu_n$ is simply totalizable to the value s if there exists a function $f(\theta)$ defined on $\sigma-e$ and possessing the following properties: (1) $f(\theta)$ is continuous at each point of $\sigma-e$; $f(\theta+0)$, $f(\theta-0)$ exist; $f(0+0)=0$, $f(1-0)=s$. (2) At each point of θ_n, $f(\theta_n+0)$, $f(\theta_n-0)$ exist, and $f(\theta_n+0)-f(\theta_n-0)=u_n$. (3) Every per-

fect set P contains a portion P' on which the total variation of $f(\theta)$ is defined and is equal to $\sum|f(\theta_n+0)-f(\theta_n-0)|$ for the points θ_n on P'. The total of the series $\sum Tu_n$ on subintervals of σ is then obtained in at most a denumerable set of operations on the numbers u_n on the subinterval by means of the function $f(\theta)$, which operations are similar to those entering into the totalization of a function. It is shown that the totalization of a series is equivalent to a simple case of the totalization of a function. *R. L. Jeffery.*

Denjoy, Arnaud. Totalisation simple des fonctions ramenée à celle des séries. C. R. Acad. Sci. Paris **210**, 73–76 (1940). [MF 1195]

This is an application to a set of functions of the idea of the above reviewed note. Let $f(x)$ be a function defined on (a, b) which is such that, if P is any perfect set on (a, b), then the points of P at which $f(x)$ is not summable over P are non-dense on P. Let $Q=Q(K)$ be the smallest perfect component of K which is such that the measure of $K-Q=0$. Form the sequence $Q_0=(a, b)$, $K_1=K(Q_0)$, $Q_1=Q(K_1)$, $\cdots$. This leads to a set $Q'=Q_0 \cdots Q_\alpha \cdots$, where α is an ordinal of the first or second class. This set is denumerable. Let it be ordered in the sequence $P_1, P_2, \cdots, P_p, \cdots$ and let ϵ_p be a sequence of positive numbers with $\sum\epsilon_p=\epsilon$. The points of Q_α not belonging to $K_{\alpha+1}$ can be grouped in portions $\omega_\alpha{}^r$ on segments $\sigma_\alpha{}^r$ with end-points $a_\alpha{}^r$, $b_\alpha{}^r$ in such a manner that (1) the $\sigma_\alpha{}^r$ interior to an interval contiguous to $K_{\alpha+1}$ have no points in common and have their points of accumulation only at the end-points of this contiguous interval; (2) for a given α the length of the largest $\sigma_\alpha{}^r$ is not greater than ϵ_p; (3) on $\omega_\alpha{}^r$ the integral of $|f|$ is not greater than ϵ_p. The intervals $\sigma_\alpha{}^r$ $(a_\alpha{}^r, b_\alpha{}^r)$ are denumerable. Designate them by ρ_n (a_n, b_n). Let $u_n(x)$ be the integral between a and x of a function equal to f on $\omega_\alpha{}^r$ and equal to zero elsewhere. If the function f is totalizable on (a, b), then, for a given x, $u_n(x)$ can be totalized by placing $u_n(x)$ at a_n, and the total of the series $(a\sum Tx)u_n(x)$ is equal to the total of f between a and x. Let $u_n(b_n)=v_n$. A necessary and sufficient condition that $(a\sum Tx)u_n(x)$ be totalizable is that $\sum Tv_n$ be totalizable. *R. L. Jeffery* (Wolfville, N. S.).

Tolstoff, G. Sur l'intégrale de Perron. Rec. Math. (Moscou) [Mat. Sbornik] N.S. **5** (47), 647–660 (1939). (French. Russian summary) [MF 1351]

The measurable function $f(x)$ is integrable P (Perron) if there exist major and minor functions to f, and if the lower bound of the major functions is equal to the upper bound of the minor functions. The author raises the question as to whether or not it is necessary to consider the infinite sets of major and minor functions. He answers this in the negative by proving: The necessary and sufficient conditions that $f(x)$ be integrable P is that (1) $f(x)$ be measurable; (2) there exist at least one major and one minor function. The necessity of (1) follows from the fact that if f is integrable P it is almost everywhere the derivative of its indefinite integral; the necessity of (2) follows from the definition. To obtain the sufficiency of the conditions, the author makes use of the fact that if f is integrable P it is integrable in the strict Denjoy sense. He shows that, if f satisfies the conditions stated, then: (1) Whatever be the closed set F, there exists a portion π of F such that f is summable over π. (2) If $f(x)$ is Denjoy integrable on every interval (α', β'), with $\alpha<\alpha'<\beta'<\beta$, then f is Denjoy integrable on (α, β). (3) Whatever be the closed set F, if f is integrable in the sense of Denjoy on the intervals contiguous to F, there exists a portion π of F such that the series of oscillations in the intervals contiguous to π is convergent.

A modification of a Perron integral has been introduced by J. C. Burkill, in which he uses approximate continuity instead of continuity, and approximate derivates in the place of derivates. The author shows that this integral is less general than the generalized Denjoy integral. He shows that, if $F(x)$ is a modified Perron integral and P any perfect set, then P contains a portion π such that the intervals (α_n, β_n) contiguous to π contain a set E_n with $mE_n/(\beta_n-\alpha_n)>1-\epsilon$, ϵ arbitrary, and such that, for any point x_n of E_n, $\sum|F(x_n)-F(\alpha_n)|$ and $\sum|F(x_n)-F(\beta_n)|$ converge. He then constructs a function which is Denjoy-integrable but which does not satisfy the foregoing. *R. L. Jeffery.*

Erim, Kerim. Ueber eine neue Definition des mehrdimensionalen Stieltjesschen Integrals. Rev. Fac. Sci. Univ. Istanbul **4**, 167–182 (1939). (German. Turkish summary) [MF 1619]

A. H. Copeland [Bull. Amer. Math. Soc. **43**, 581–588 (1937)] gave a new definition of the Riemann-Stieltjes integral of a function $f(x)$ based on arithmetic means of ordinates taken over a sequence of points equidistributed with respect to the measure generated by the integrator function. The author carries through the discussion of the generalization to two and three dimensions. One of the main points established by Copeland is that his definition is actually wider than the classical one: this aspect is not discussed in the present paper. *J. A. Clarkson.*

Jeffery, R. L. Functions of bounded variation and non-absolutely convergent integrals in two or more dimensions. Duke Math. J. **5**, 753–774 (1939). [MF 816]

Let the function $f(x, y)$ be single valued on the rectangle $R=(0, 0; a, b)$, and let $s_n(x, y)$ be a sequence of summable functions tending to f. Then $F=\lim\int_0^y\int_0^x s_n dxdy$ $(n\rightarrow\infty)$ is in class V_2 on R; if moreover $\int_e s_n dxdy$ is bounded in n and e, then F is in class V_1 on R. Let F be in class V_1 on R; the associated function f is measurable and summable; F is in class H (Hardy-Krause); F is the difference of two monotone functions; and F is totally differentiable (and continuous) almost everywhere. If F is completely additive on R, then $F(R)=\int_R F'dxdy$; if E is any measurable set on R, then a necessary and sufficient condition that $F(e)=\int_e fdxdy$ for every measurable subset e of E is that F be completely additive over R. Let F be in class V_2 on R; if F is of class V_1 over a measurable subset E of R, then f is summable on E; if the points of any closed set E on R at which F is not of class V_1 and completely (absolutely) additive are non-dense on E, then a specialized derivate (the derivate) of F exists almost everywhere on R and is equal to f. Let $F(w)$ be a continuous additive function of intervals defined on R. If the derivate F' is finite at each point of R, then it is possible to determine F in at most a denumerable set of operations on F'; if the strong derivate F_s' is finite at each point of R, then F can be determined from F_s' by a process which is the analogue of the Denjoy process. This process for inverting the strong derivate may be made the basis of a constructive definition of a non-absolutely convergent integral of a measurable function $f(x, y)$.

J. G. van der Corput (Groningen).

Calkin, J. W. Functions of several variables and absolute continuity, I. Duke Math. J. **6**, 170–186 (1940). [MF 1551]

The author defines and discusses the relations between

classes of functions $\mathfrak{P}$, $\mathfrak{P}'$, $\mathfrak{P}''$, which are akin to the "potential functions of their generalized derivatives" of G. C. Evans [Rice Inst. Pamphlet 7, 252–329 (1920)] and to the absolutely continuous functions of Tonelli. A function $f(x_1, x_2, \cdots, x_n)$ (denoted also by $f(x)$ or $f(x_k', x_k)$, where x is the variable $(x_1, x_2, \cdots, x_n)$ and x_k' the variable $(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)$) is said to be E.A.C. (essentially absolutely continuous) in x_k over a region G if it is integrable over G and there is a function $g_k(x)$ such that, for almost all cells contained in (a, b),

$$(1)\qquad \int_{a_k'}^{b_k'} f(x_k', b_k) - f(x_k', a_k)dx_k' = \int_a^b g_k(x)dx.$$

The generalized derivative $D_{x_k}f$ is defined to be $g_k(x)$ p.p. A function $f(x)$ is said to be L.A.C. (linearly absolutely continuous) in x_k over G if (i) $f(x)$ is integrable over G, (ii) $f(x)$ is absolutely continuous in x_k for almost all values of x_k', (iii) $\partial f/\partial x_k = g_k$ satisfies (1).

The function is of class (i) $\mathfrak{P}$ if it is E.A.C. for every x_k, (ii) $\mathfrak{P}'$ if it is L.A.C. for every x_k and (iii) $\mathfrak{P}''$ if it is of class $\mathfrak{P}$ and also continuous. Corresponding classes $\mathfrak{P}_\alpha$, $\mathfrak{P}_\alpha'$ are defined in which $|f(x)|^\alpha$ and $|D_{x_k}f|^\alpha$ or $|\partial f/\partial x_k|^\alpha$ are also integrable. Clearly $\mathfrak{P}'' \supset \mathfrak{P}' \supset \mathfrak{P}$, and it is shown that every function of class $\mathfrak{P}$ is equivalent to a function of class $\mathfrak{P}'$. Moreover, a necessary and sufficient condition that an integrable function be of class $\mathfrak{P}$ (or $\mathfrak{P}_\alpha$) is that there exist integrable functions $g_1, \cdots, g_n$ and a sequence $\{f_p(x)\}$ of functions satisfying a Lipschitz or similar condition such that, in the mean (or in mean αth power), $f_p(x) \to f(x)$ and $\partial f_p/\partial x_k \to g_k$. If $f(x)$ is of class $\mathfrak{P}''$, then $\{f_p(x)\}$ converges uniformly to $f(x)$. Finally it is shown that, with a suitably chosen norm, the space of functions of class $\mathfrak{P}_\alpha$ is a Banach space. *U. S. Haslam-Jones* (Oxford).

Morrey, C. B., Jr. Functions of several variables and absolute continuity, II. Duke Math. J. 6, 187–215 (1940). [MF 1552]

Further properties of functions of classes $\mathfrak{P}$, $\mathfrak{P}'$ and $\mathfrak{P}''$ defined by J. W. Calkin [see above review] are here obtained. It is first shown that, if $f(x)$ is of class $\mathfrak{P}$ on a set S and $g(y) = f(x(y))$ is its transform by a transformation $x = x(y)$ of class K (that is, which is 1-1 and continuous and such that $x(y)$ and its inverse satisfy uniform Lipschitz conditions), then $g(y)$ is also of class $\mathfrak{P}$, and $D_{x_k}f$ is related to $D_{y_k}g$ by the usual formula of partial differentiation. Also any function of $\mathfrak{P}$ is equivalent to a function of $\mathfrak{P}'$ which remains of class $\mathfrak{P}'$ under a transformation of class K. Theorems are established concerning strong convergence in $\mathfrak{P}_\alpha$ and concerning the boundary values of functions of $\mathfrak{P}_\alpha$ in an open region of a restricted type. It is shown that, if $z(x)$ belongs to $\mathfrak{P}_\alpha$ in a region G, it is possible to choose a sequence $\{z_p(x)\}$ of functions of $\mathfrak{P}_\alpha$ which converges strongly to $z(x)$ in G and to a function $\varphi(x)$ of L_α on G^* (the boundary of G); also that the relation between $\varphi(x)$ and $z(x)$ is invariant under transformations of class K. The values of $\varphi(x)$ are called the boundary values of $z(x)$. The analogues of standard theorems concerning weak convergence in L_α are proved for weak convergence in $\mathfrak{P}_\alpha$; and it is further shown that, if $\{z_p(x)\}$ converges weakly in $\mathfrak{P}_\alpha$ to $z(x)$ over a region G, then the sequence converges strongly in L_α to $z(x)$ in each closed cell of G. The paper concludes with a discussion of functions of class $\mathfrak{P}_\alpha$ in an arbitrary bounded region, including the boundary values.

U. S. Haslam-Jones (Oxford).

Kempisty, Stefan. Sur l'aire des surfaces courbes continues. Fund. Math. 33, 34–41 (1939). [MF 1722]

Let S be a surface represented on the square $Q: 0 \leqq u, v \leqq 1$, by functions $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ which are continuous on Q and possess partial derivatives almost everywhere on Q. Let $R: a \leqq x \leqq a+h$, $b \leqq y \leqq b+k$ be a rectangle in Q, let $L(R)$ denote the Lebesgue area of the part of S corresponding to R, and let $F(R)$ be the sum of the areas of the two triangles whose vertices are the points of S corresponding to the points (a, b), $(a+h, b)$ and $(a+h, b+k)$ and (a, b), $(a, b+h)$ and $(a+h, b+k)$. Such a rectangle is semi-regular if $\frac{1}{2} \leqq h/k \leqq 2$. Let $\overline{\int}_Q F$ and $\underline{\int}_Q F$ denote the upper and lower Burkill integral allowing only semi-regular rectangles in the subdivisions and $\bar{D}F$ and $\underline{D}F$ denote the upper and lower limits of $F(R)/m(R)$ as $R \to (u, v)$, always containing (u, v) and being semi-regular; $\int_Q F$ and DF are defined in the usual way. $F(R)$ is said to be semi-regularly absolutely continuous if the ordinary definition is satisfied only for sets of semi-regular rectangles.

Under the above hypotheses on x, y, z, the following results are proved: Theorem 1: If $\overline{\int}_Q F < \infty$, then, almost everywhere, we have

$$\underline{D}F = DL = (X^2 + Y^2 + Z^2)^{\frac{1}{2}},$$

where X, Y, Z denote the Jacobians $X = \partial(y, z)/\partial(u, v)$, etc. Theorem 2: If F is semi-regularly absolutely continuous, then

$$\underline{\int}_Q F = L(Q) = \iint_Q (X^2 + Y^2 + Z^2)^{\frac{1}{2}} dudv.$$

Theorem 3: If, in addition to the hypotheses of theorem 2, x, y and z possess total differentials in the sense of Stolz almost everywhere, then

$$\int_Q F = L(Q) = \iint_Q (X^2 + Y^2 + Z^2)^{\frac{1}{2}} dudv.$$

C. B. Morrey (Berkeley, Calif.).

Appert, Antoine. Mesure dans l'espace à une infinité de coordonnées. Revista Ci., Lima 41, 297–308 (1939). [MF 1639]

If R denotes the space of all real sequences $(x_1, x_2, \cdots)$, $-\infty < x_i < +\infty$, and I a subset of R consisting of all points $(x_1, x_2, \cdots)$ for which $a_i \leqq x_i \leqq b_i$, $i = 1, 2, \cdots$ $(-\infty < a_i \leqq b_i < +\infty)$, then the set (or measure) function defined for I by

$$|I| = \lim_{n\to\infty} \prod_{i=1}^{n} (b_i - a_i)$$

may be extended to an absolutely additive measure over the Borel field determined by the family of subsets I. This extension is obtained by the standard procedure of defining the outer measure of a subset E of R as the greatest lower bound of the sums $\sum_k |I^{(k)}|$, where E is contained in the (logical) sum $\sum_k I^{(k)}$, and then defining measurable sets in the Carathéodory manner. *P. Hartman.*

van Kampen, E. R. Infinite product measures and infinite convolutions. Amer. J. Math. 62, 417–448 (1940). [MF 1775]

Let X_n be an infinite sequence of sets each carrying a normalized Lebesgue measure μ_n and let $X = \Pi X_n$ denote the infinite combinatorial product. After a "product measure" is introduced in X the author develops in a systematic way the convergence and distribution theory of sums $\sum f_n(x_n)$

($x_n \epsilon X_n$, f_n measurable) considered on X. The known results of Kolmogoroff, Lévy, Paley, Zygmund, Wintner and others are presented here in a comprehensive and intelligible way with highly simplified and unified proofs. Many new results are also obtained. The paper ends with an extensive bibliography of the subject. *M. Kac* (Ithaca, N. Y.).

Theory of Functions of Complex Variables

Valeiras, Antonio. On a question in geometry relative to power series. An. Soc. Ci. Argentina **128**, 217–221 (1939). (Spanish) [MF 1076]

The author makes geometrical remarks about the expression

$$w=\frac{|1-z|}{1-|z|}, \qquad z=x+iy,$$

which appears in the proof of Abel-Stolz theorem.
A. González Domínguez (Providence, R. I.).

Serbin, H. Weierstrass preparation theorem. Bull. Amer. Math. Soc. **46**, 168 (1940). [MF 1265]

A simple proof by induction of a formal result from which the Weierstrass preparation theorem may be deduced.
P. Franklin (Cambridge, Mass.).

Edrei, Albert. Sur les déterminants récurrents et les singularités d'une fonction donnée par son développement de Taylor. Compositio Math. **7**, 20–88 (1939). [MF 368]

Consider the power series

$$\frac{a_0}{z}+\frac{a_1}{z^2}+\cdots+\frac{a_n}{z^{n+1}}+\cdots=f(z) \tag{1}$$

and form from its coefficients the determinants

$$A_k^{(n)}=\begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+n-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k+n-1} & a_{k+n} & \cdots & a_{k+2n-2} \end{vmatrix}. \tag{2}$$

Many connections between the determinants $A_k^{(n)}$ and the analytic nature of the function $f(z)$ defined by the series (1) are known. In particular, if the function defined by the analytic continuation of the series (1) is single-valued in its region of definition and if E denotes the set of singularities of $f(z)$, it is clear that E is closed and bounded and that its complement is connected. Under these conditions, it was proved by Pólya [Math. Ann. **17**, 228–249 (1923)] that

$$\overline{\lim_{n\to\infty}}\, |A_0^{(n)}|^{1/(n(n-1))} \leqq \tau, \tag{3}$$

where τ is the capacity of the set E. Furthermore, R. Wilson [Proc. London Math. Soc. (2) **39**, 363–371 (1935)] showed that if E consists only of poles and a single essential singularity of finite order ρ which is either isolated or a limit point of poles, then

$$\overline{\lim_{n\to\infty}}\, |A_0^{(n)}|^{1/(n^2 \log n)} \leqq e^{-(1/\rho)}. \tag{4}$$

The author extends the study of problems of this type. Of the many results of the paper we shall mention only a few. The author obtains a generalization of Wilson's theorem to the case that $f(z)$ possesses a finite number of essential singularities of finite order. Furthermore, from the study of analytic functions represented by certain continued fractions, the author shows that under the conditions of Pólya's and Wilson's theorems the inequalities asserted may not, in general, be replaced by equalities. In fact, even a complete knowledge of the sequence $\{A_0^{(n)}\}$ does not enable one to make any precise assertions concerning the nature of the singularities of $f(z)$. On the other hand, the knowledge of the sequences $\{A_0^{(n)}\}$ and $\{A_1^{(n)}\}$ is equivalent to the knowledge of the whole expansion (1). Furthermore, if $\{l_n\}$ and $\{k_n\}$ are two sequences of arbitrary complex numbers with all $k_n \neq 0$ and α an arbitrary complex number such that

$$\overline{\lim_{n\to\infty}}\, |l_n-\alpha|^{1/\log n} \leqq \overline{\lim_{n\to\infty}}\, |k_1^n k_2^{n-1} \cdots k_{n-1}^2 k_n^1|^{1/(n^2 \log n)}$$
$$= \overline{\lim_{n\to\infty}}\, |k_n|^{1/(2 \log n)} = e^{-(1/\rho)},$$

where $0 \leqq \rho < +\infty$, the continued fraction

$$\frac{k_1|}{|z-l_1}-\frac{k_2|}{|z-l_2}-\cdots-\frac{k_n|}{|z-l_n}-\cdots$$

represents a single-valued function whose only singularities are poles with the exception of a single essential singularity at the point $z=\alpha$ of order ρ. If (1) denotes the expansion of this function at $z=\infty$, the inequality (4) becomes an equality. The author, furthermore, shows that Pólya's inequality (3) becomes an equality for certain functions of the type

$$f(z)=\int_\alpha^\beta \frac{\varphi(x)dx}{x-z}.$$

W. Seidel (Rochester, N. Y.).

Biggeri, Carlos. On a theorem on the singular points of analytical functions defined by general Dirichlet series. Bol. Mat. **13**, 8–10 (1940). (Spanish) [MF 1725]

The author reproduces the proof of an equality [formula 7 of the present note] as given by him in two previous notes [cf. these Rev. **1**, 113 (1940)] and makes two observations about it. *A. González Domínguez* (Providence, R. I.).

Walsh, J. L. Note on the curvature of orthogonal trajectories of level curves of Green's function. III. Bull. Amer. Math. Soc. **46**, 101–108 (1940). [MF 1254]

Let R be a simply connected domain in the complex w-plane, O an interior point of R and D the "conjugate" of O with respect to R [this concept was previously defined by the author in geometrical terms as well as in terms of the map function of R; see Amer. Math. Monthly **42**, 1 (1935)]. Let $\{w_n\}$ be a sequence of points of R approaching a boundary point w_0 of R; the author investigates the convergence of the conjugates $\{w_n'\}$. They approach w_0 provided R and $\{w_n\}$ satisfy certain conditions. However, a domain R and a sequence $\{w_n\}$ can be constructed such that $w_n' \to \infty$. *G. Szegö* (Stanford University, Calif.).

Schiffer, Menahem. Sur la variation de la fonction de Green de domaines plans quelconques. C. R. Acad. Sci. Paris **209**, 980–982 (1939). [MF 1199]

Let $\mathcal{E}$ be a closed set in the plane and $\mathcal{E}^*$ the transform of $\mathcal{E}$ by the function $z^*=z+\epsilon e^{i\varphi}(z-z_0)^{-1}$, where $\epsilon>0$, $0 \leqq \varphi \leqq 2\pi$, and $z_0 \notin \mathcal{E}$. If ϵ is small enough, z^* is uniform on $\mathcal{E}$. Let $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}}^*$ be the infinite regions exterior to $\mathcal{E}$ and $\mathcal{E}^*$, respectively, and $g(\zeta, z)$ and $g^*(\zeta, z)$ be the corresponding Green's

functions. The author undertakes the problem of finding in terms of ϵ the variation of the Green's function in passing from $\bar{S}$ to $\bar{S}^*$. Let $q(\zeta, z)$ be the analytic function whose real part is $g(\zeta, z)$, and let $q'(x, y)$ be $\partial/\partial x\, q(x, y)$. The following formula for the variation of the Green's function is obtained:

$$g(z, \infty)-g^*(z, \infty) \\ =\Re\{\epsilon e^{i\phi}[q'(z_0, z)q'(z_0, \infty)-q'(z, \infty)(z-z_0)^{-1}]\}+O(\epsilon^2).$$

The method of proof involves the use of the various finite diameters of Fekete and the polynomials associated with them. *J. W. Green* (Rochester, N. Y.).

Wittich, Hans. Über die konforme Abbildung einer Klasse Riemannscher Flächen. Math. Z. **45**, 642–668 (1939). [MF 1406]

Let W denote a Riemann surface with the following properties: W is spread out over the w-plane and is simply connected. The number of the sheets of W is infinite and all the branch-points of W are located over a finite number of points of the w-plane. If then W is mapped, according to the fundamental theorem of the theory of uniformization, upon a disc $|z|<R\leqq\infty$, there arises the fundamental problem of determining the type of W, that is, the problem of deciding which one of the cases $R<\infty$ and $R=\infty$ occurs for a given W. The purpose of the paper is to contribute a new sufficient condition for $R=\infty$, in which case W is said to be of the parabolic type. The method is analogous to that used by Nevanlinna and Ahlfors [see the book of R. Nevanlinna, Eindeutige analytische Funktionen, Berlin, 1936]. The main result is stated in terms of the line complex S which Nevanlinna associated with W and more exactly in terms of the number $\sigma(n)$ of the boundary vertices of the sub-complexes $S(n)$ of S, as follows: If the series $\sum[1/\sigma(n)]$ is divergent, then W is of the parabolic type. In conclusion, it is shown first that various former results are included in this theorem, and second that even in simple cases the present criterion might fail to yield information as to the type of W. *T. Radó* (Columbus, Ohio).

Brödel, Walter. Fortgesetzte Untersuchungen über Deformationsklassen bei mehrdeutigen topologischen Abbildungen. Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 229–260 (1939). [MF 1189]

According to the Lüroth-Clebsch theorem, the algebraic Riemann surfaces of a fixed genus p and a fixed number m of sheets constitute a continuum; that is, any surface of this type can be mapped on any other by a continuous shifting of its branch points (whereby, in particular, branch points of higher order can be resolved into simple ones, and conversely). Accordingly, algebraic Riemann surfaces with movable branch points have as invariants only m, p. For closed, orientable basic surfaces F of genus 1 (that is, for the torus), the author now investigates covering surfaces Φ of the same sort as those of algebraic functions, namely, surfaces which are unbounded, closed, and with a finite number of sheets and branch points. The branch points are considered to be movable. A system of invariants is developed for surfaces Φ, such that the systems coincide for two surfaces if and only if the surfaces can be mapped continuously on each other. Surfaces of equal genus p and equal number m of sheets now fall into several continua.

Let F_0 be the most extensive unbranched covering surface for F, say m_0-sheeted, such that Φ is a covering surface for F_0, and let Φ be n-sheeted relative to F_0, $n=m/m_0$. If $n\leqq p$, then Φ is built up of n copies of F_0 in such a way that the first and second sheets are joined along $p-n+1$ separate cuts, while the second and third, the third and fourth, etc., are joined along single cuts. If $n>p$, then Φ can be built up of p closed unbranched covering surfaces $F_1, \cdots, F_p$ of F_0, each two successive surfaces being joined along a single cut. Necessary and sufficient conditions for two surfaces Φ_1, Φ_2 to be deformable into each other by continuous shifting of the branch points are equality of genus and number of sheets, and in addition the agreement of their surfaces F_i. *E. F. Beckenbach* (Houston, Tex.).

Hibbert, Lucien. Sur les faisceaux de courbes V=const. des fonctions entières. C. R. Acad. Sci. Paris **209**, 783–786 (1939). [MF 865]

L'auteur utilise les définitions de ses notes précédentes [C. R. Acad. Sci. Paris **205**, 1121 (1937); **207**, 891, 961 (1938)], $f(z)$ étant une fonction holomorphe de z, sauf en A (fini) qui est point essentiel; il étudie les propriétés des courbes $V=\arg f(z)=$const., et $R=|f(z)|=$const.; ces courbes pouvant être décrites dans le sens des R croissants ($\vec{V}$) ou des R décroissants ($\overleftarrow{V}$), et de même pour les R. Il fait cette étude directement dans le plan des z sans utiliser la surface de Riemann définie par les valeurs de $f(z)$ comme l'ont fait la plupart des auteurs antérieurs à la suite d'Iversen [Thèse, Helsingfors, 1914], notamment Shimizu [Jap. J. Math. **8**, 175–304 (1931)]. Il considère ici les courbes $\vec{V}$ coupant un arc de courbe R=const.; il rectifie et complète des résultats antérieurs en étudiant le cas où celles de ces courbes qui ne sont pas normales (c'est-à-dire, se ramifient avant d'arriver en A ou arrivent en A avec R fini) sont en nombre infini. Il déduit de ses propositions antérieures que les courbes normales coupent l'arc considéré en des points denses sur cet arc [Proposition déduite en général d'un théorème de Gross; voir Shimizu, loc. cit.]. L'auteur rappelle un travail du réferent [Bull. Sci. Math. **63**, 132 (1939)] dont le but était précisément de signaler diverses circonstances qui se rencontrent dans ce genre de questions. *G. Valiron* (Paris).

Hibbert, Lucien. Propriétés des faisceaux $f(\overleftarrow{V}, \vec{bc})$ de parcours négatifs V des fonctions entières. C. R. Acad. Sci. Paris **210**, 35–37 (1940). [MF 1243]

The author considers entire functions $f(z)=Re^{iV}$, and their asymptotic values on the curves R=constant and V=constant. A positive (negative) V-circuit, denoted by $\vec{V}$ ($\overleftarrow{V}$), is defined to be a circuit, not necessarily closed, over a curve V=constant, in the direction of increasing (decreasing) R. Circuits $\vec{R}$ and $\overleftarrow{R}$ are defined analogously. These curves form a network in the plane with singularities at the zeros of R and of $f'(z)$, and the author has considered its properties in previous papers [C. R. Acad. Sci. Paris **205**, 1121; **207**, 891 and 961]. By the topological properties of the net of curves, the following theorem is obtained: Let $w=f(z)$ be entire; there exist infinitely many negative V-circuits $\overleftarrow{V}$ along which $f(z)$ tends to the value zero. The theorem can also be stated in terms of any value w_0 instead of zero by making use of the entire function $w_1=f(z)-w_0$. *J. W. Green* (Rochester, N. Y.).

Töpfer, Hans. Über die Iteration der ganzen transzendenten Funktionen, insbesondere von sin z und cos z. Math. Ann. **117**, 65–84 (1939). [MF 1383]

L'auteur apporte d'abord quelques compléments intéressants aux théorèmes généraux de Fatou [Acta Math. **47**, 1926]. Soient $f(z)$ la fonction entière que l'on itère, et F l'ensemble des points en lesquels la suite des itérées $f_n(z)$

$[f_2=f(f), f_n=f_{n-1}(f)]$ n'est pas normale. Si F ne comprend pas tout le plan, on appelle G tout domaine formé de points extérieurs à F et dont la frontière appartient à F. L'auteur établit que: I. S'il existe un domaine G_0 admettant ∞ pour point frontière, tout domaine G_1 distinct de G_0 est simplement connexe; si G_0 est multiplement connexe, il est complétement invariant. Il s'ensuit, comme dans le cas de l'itération des fractions rationnelles, qu'il existe au plus deux domaines complétement invariants. II. L'ensemble F ne contient aucun arc de courbe de Jordan isolé. III. Si a est un point périodique répulsif, non remarquable au sens de Fatou, si a est point frontière accessible d'un domaine invariant G simplement connexe, et si le multiplicateur de a n'est pas un nombre réel positif, sur tout chemin Γ de G tendant vers a, l'argument de $z-a$ tend vers $+\infty$ ou vers $-\infty$. L'auteur étudie ensuite les propriétés de l'ensemble F dans le cas de l'itération de $\sin z$ et de $\cos z$. Il complète aussi les propositions énoncées sans démonstration par Fatou à la fin de son mémoire et relatives aux propriétés de F dans l'itération de e^z. *G. Valiron* (Paris).

Boas, R. P., Jr. Some uniqueness theorems for entire functions. Amer. J. Math. **62**, 319–324 (1940). [MF 1767]

The author proves the following theorem as a special case of a more general theorem: If $f(z)$ is an entire function of exponential type such that

$$f(iy)=O(e^{k|y|}),\qquad k<\pi,$$
$$f(x)=O(e^{l|x|}),$$

and

$$l<\frac{\pi}{2}\cot\frac{\pi}{2}\left(1-\frac{1}{2q-1}\right),$$

where q is an integer, then $f(z)\equiv 0$ if

$$f(2n)=f^{(2q-1)}(2n)=0,\qquad n=0, \pm 1, \pm 2, \cdots.$$

In the general theorem of which this is a special case, $f^{(2q-1)}(2n)=0$ is replaced by the condition that a linear combination of derivatives of $f(z)$ at $z=2n$ vanish, where the linear combination satisfies certain requirements. The order of differentiation involved in this linear combination of derivatives need not be finite. *N. Levinson.*

Boas, R. P., Jr. Entire functions bounded on a line. Duke Math. J. **6**, 148–169 (1940). [MF 1550]

Using an absolutely convergent interpolation series for entire functions of exponential type and results of Wiener and Paley on non-harmonic Fourier series, the author generalizes results of Cartwright, Pólya and Plancherel and obtains a proof for a theorem of the reviewer. One of the theorems is: If $f(z)$ is an entire function of exponential type $k<\pi$ and if $|f(\lambda_n)|\leqq K$ $(n=0, \pm 1, \pm 2, \cdots)$, where λ_n are real and $|\lambda_n-n|\leqq L<1/(2\pi^2)$, then

$$|f(x)|\leqq\frac{A(k)K}{1-\pi(2L)^{\frac{1}{2}}},\qquad -\infty<x<\infty.$$

The case $\lambda_n=n$ is a result of Cartwright. Another theorem proved is: Let $\beta(x)$ be a non-decreasing function defined on $(-\infty, \infty)$ such that $\beta(x+1)-\beta(x)\leqq B<\infty$. Let $\gamma(t)$ be a non-decreasing function defined in $(-\delta, \delta)$, $\delta>0$. If $f(z)$ is an entire function of exponential type $k<\pi$, and $p\geqq 1$, then

$$\int_{-\infty}^{\infty}|f(x)|^p d\beta(x)\leqq\frac{B\{C(k, \delta)\}^p}{\gamma(\delta)-\gamma(-\delta)}\sum_{n=-\infty}^{\infty}\int_{-\delta}^{\delta}|f(n+t)|^p d\gamma(t).$$

In particular, if $\beta(x)=x$ and $\gamma(t)=\operatorname{sgn} t$, the above inequality becomes

$$\int_{-\infty}^{\infty}|f(x)|^p dx\leqq\{C(k, 0)\}^p\sum_{-\infty}^{\infty}|f(n)|^p,$$

which is due to Plancherel and Pólya. *N. Levinson.*

Wright, E. M. The asymptotic expansion of integral functions defined by Taylor series. Philos. Trans. Roy. Soc. London, Ser. A. **238**, 423–451 (1940). [MF 1279]

L'auteur considère les fonctions entières de la forme

$$f(x)=c_0+c_1x+\cdots+c_nx^n+\cdots$$

où

$$c_n=\frac{\phi(n)}{\Gamma(kn+\beta)},$$

k et β sont des constantes réelles ou complexes, $\Re(k)>0$; $\phi(t)$ est une fonction régulière développable suivant les puissances décroissantes, entières ou non, de t, pour $\Re(kt)>h$. Il donne des développements asymptotiques, de type exponentiel, de $f(x)$ valables pour les grandes valeurs de x, lorsque $\Re(1/k)<\frac{1}{2}$; tandis que, si $\Re(1/k)>\frac{1}{2}$, on a un développement de type exponentiel pour les grands x intérieurs à un domaine limité par deux branches de spirales, un développement de type algébrique lorsque x est dans le domaine complémentaire, un développement mixte dans le voisinage des spirales. Dans le cas $\Re(1/k)=\frac{1}{2}$, les deux spirales coïncident; le développement dans le domaine unique est du type exponentiel. Les développements du type exponentiel comportent une expression de la forme

$$ye^y\sum A_m y^{-m}$$

et un reste, y étant une fonction convenable de x. L'auteur emploie principalement les méthodes de Watson [Rend. Circ. Mat. Palermo **34** (1912); Trans. Cambridge Philos. Soc. **22** (1913)]; ses résultats contiennent et complètent ceux de cet auteur, ainsi que ceux de Wiman sur les fonctions de Mittag-Leffler, de Hardy et de Barnes. *G. Valiron* (Paris).

Tchakaloff, Lubomir. Sur quelques propriétés des développements de Taylor d'une certaine classe de fonctions. Proc. Benares Math. Soc. **1**, 25–33 (1939). [MF 1528]

Let C be the class of functions $f(x)$ which are analytic in the half plane $\Re(x)<0$ and in a neighborhood of the origin, and which, together with all their derivatives, are positive on the negative real axis. The author studies the zeros of polynomial sections of functions of class C. For example, if $f(x)=c_0+c_1x+\cdots$, it is shown that the polynomial $c_n+c_{n+1}x+\cdots+c_{n+p}x^p$ can have at most one real zero, which (if it exists) is negative and simple. More general results of a similar nature are also obtained. *A. C. Schaeffer* (Palo Alto, Calif.).

Montel, Paul. Sur les valeurs des fonctions holomorphes. C. R. Acad. Sci. Paris **209**, 963–967 (1939). [MF 1196]

If, in $|z|\leqq 1$, $f(z)$ is holomorphic and $f(z_1)=a$, $f(z_2)=b$, then there exists a positive constant $\delta=\delta(a, b, \max|f(z)|)$ such that the non-euclidean distance between z_1 and z_2 is not less than δ. The author determines the exact value of δ for a significant class of functions, and exhibits the function for which the minimum is attained. The functions considered are functions

$$f(z)=a+a_1z+a_2z^2+\cdots+a_nz^n+\cdots,$$

holomorphic in $|z|\leqq 1$ and satisfying

$$\sum_{n=1}^{\infty}|a_n|^{\alpha}b_n^{\alpha}\leqq R^{\alpha},$$

where R, α, and the b_n are constants with $R>0$, $\alpha>1$, $b_n\geqq 0$, $\overline{\lim}\,(b_n)^{1/n}\geqq 1$. For example, if $\alpha=2$, $b_n=1$, the condition imposed on $f(z)$ is that the mean of order 2 of $|f(z)-a|$ is bounded on $|z|=1$. *E. F. Beckenbach* (Houston, Tex.).

Suñer y Balaguer, F. On a theorem of Professor Picard. Revista Mat. Hisp.-Amer. (3) **1**, 27–28 (1939). (Spanish) [MF 1671]

The author proves the "little" Picard theorem by applying the theory of normal families in a new way. If $f(z)$ is meromorphic, $f(c)\neq f(0)$, and $f(z)$ has three exceptional values, the author considers the normal family $\{f(g_n(z))\}$, where $\{g_n(z)\}$ is a sequence of polynomials which converges to zero when $z=1$ and to c when $z=2$. A contradiction results at once (without appeal to Liouville's theorem). *R. P. Boas, Jr.* (Durham, N. C.).

Milloux, Henri. Sur la théorie des défauts. C. R. Acad. Sci. Paris **210**, 38–39 (1940). [MF 1244]

Uebertragung auf meromorphe Funktionen und Verschärfung im Sinne der Defekt-Theorie von R. Nevanlinna des Satzes betreffend ganze Funktionen, wonach der Picard'sche Ausnahmewert der ersten Ableitung einer ganzen Funktion 0 sein muss, wenn die Funktion selbst einen Picard'schen Ausnahmewert besitzt. *W. Saxer* (Zürich).

Herzig, Alfred. Die Winkelderivierte und das Poisson-Stieltjes-Integral. Math. Z. **46**, 129–156 (1940). [MF 1486]

The paper starts out with a proof of the Julia-Wolff-Carathéodory-Landau-Valiron theorem on the angular derivative (ang. d.): Let $F(z)$ be analytic and $|F(z)|\leqq 1$ for $|z|<1$. Then the ang. d.

$$\lim_{z\to 1}\frac{F(z)-1}{z-1}=D$$

for non-tangential approach exists, and $0\leqq D\leqq+\infty$. If D is finite,

$$(*)\qquad \frac{|1-F(z)|^2}{1-|F(z)|^2}\leqq D\frac{|1-z|^2}{1-|z|^2}$$

and $\lim_{z\to 1}F'(z)=D$ for non-tangential approach. The equality in $(*)$ holds only if $F(z)$ is a certain rational function; $D=0$ only if $F(z)\equiv 1$. The proof given by the author is based upon the representation of the function

$$f(z)=\frac{1+F(z)}{1-F(z)},$$

whose real part is not less than 0 in $|z|<1$, by the Stieltjes-Poisson Integral:

$$(1)\qquad f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\alpha(\theta)+i\Im f(0),$$

$$\alpha(\theta)\text{ non-decreasing; } -\pi\leqq\theta\leqq\pi.$$

This proof, however, was given previously by R. Nevanlinna [C. R. Acad. Sci. Paris **188**, 1027–1029 (1929); the author does not seem to have been aware of this note]. Beyond the fact that $D\geqq 0$, the author shows that, if

$F(z)=A_nz^n+A_{n+1}z^{n+1}+\cdots$ for $|z|<1$,

$$D\geqq n+\frac{|1-A_n|^2}{1-|A_n|^2},$$

the equality holding only for a certain rational function. Further remarks on the ang. d. conclude the first part of the paper. The second part begins with a study of the coefficients in the expansion of a function $f(z)=1+\sum_{n=1}^{\infty}a_nz^n$ with $\Re f(z)\geqq 0$ in $|z|<1$. From the representation (1) one finds $|a_n|\leqq 2$. If $|a_n|=2$ for some n then, it is shown, $\alpha(\theta)$ is a step function with n jumps at points $\theta=\theta_k$ $(k=1,2,\cdots,n)$, the saltus at θ_k being μ_k, $\sum_{k=1}^{n}\mu_k=1$, and, therefore,

$$f(z)=\sum_{k=1}^{n}\mu_k\frac{e^{i\theta_k}+z}{e^{i\theta_k}-z}.$$

As the author remarks, this statement is contained in Carathéodory's results on functions with nonnegative real part [Math. Ann. **64**, 95–115 (1907). Compare also G. Herglotz, Ber. Verh. Sächs. Akad. Wiss. Leipzig **63**, 501–511 (1911)]. As one of several applications, it is shown that a necessary and sufficient condition for a function $F(z)$ to satisfy the hypotheses of Schwarz's Lemma and to have continuous boundary values of absolute value 1 on $|z|=1$ is that $\alpha(\theta)$ in the representation (1) of

$$f(z)=\frac{1+F(z)}{1-F(z)}$$

be a step function with the "total" saltus equal to 1. Finally, extending the theorem on the ang. d., the author proves that if in (1) $\alpha(\theta)=O(|\theta|^{1-\lambda})$ as $\theta\to 0$, $0<\lambda<1$, then

$$(1-F(z))^{-1}=O\left(\frac{1}{|1-z|^{\lambda}}\right)$$

for non-tangential approach $z\to 1$. As a corollary, an extension of a classical theorem on the Poisson integral is obtained: If $u(\theta)$ is L-integrable, $-\pi\leqq\theta\leqq\pi$, and $u(\theta)=O(|\theta|^{-\lambda})$ as $\theta\to 0$, $0<\lambda<1$, then

$$U(z)=U(re^{i\alpha})$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(\theta)\frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)}d\theta=O\{(1-r)^{-\lambda}\}$$

for non-tangential approach $z\to 1$. *S. E. Warschawski.*

Blanc, Charles. Une interprétation élémentaire des théorèmes fondamentaux de M. Nevanlinna. Comment. Math. Helv. **12**, 153–163 (1939–40). [MF 1048]

En vue d'une extension de la théorie des fonctions méromorphes, l'auteur introduit sur un réseau constitué par une suite de cercles concentriques et p rayons, des fonctions définies aux sommets et diverses notions analogues à l'harmonicité, pôles, zéros, fonctions de Green et de Nevanlinna avec des propriétés correspondantes. *M. Brelot.*

Sidon, S. Über Potenzreihen mit monotoner Koeffizientenfolge. Acta Litt. Sci. Szeged **9**, 244–246 (1940). [MF 1229]

Let $a_1, a_2, \cdots, a_n$ be given complex numbers, $A_1, A_2, \cdots, A_n$ the corresponding points of the Gaussian number plane with origin O. Then the equation $\sum_{k=1}^{n}p_ka_k=0$ in the unknown p_k has solutions for which $p_k\geqq 0$, $k=1,2,\cdots,n$, $\sum p_k>0$, when and only when the rays $OA_1, OA_2, \cdots, OA_n$ do not lie within the one and the same half-plane. The

author shows how the above lemma of A. J. Kempner [Math. Ann. **85**, 49–59 (1922)] effectively furnishes a simple proof of the well-known Eneström-Kakeya theorem that if $c_k > c_{k+1}$, $k=0, 1, \cdots, n-1$, $c_n \geqq 0$, then $\sum_0^n c_k z^k \neq 0$, $|z|<1$. The further information is obtained that the zeros of this class of polynomials fill out the set of values z for which the inequalities $\Re\{(1-z^k)e^{i\alpha}\}>0$, $k=1, 2, \cdots, n+1$, are fulfilled simultaneously for no real α. L. Fejér [Acta Litt. Sci. Szeged **8**, 89–115 (1936)] has shown that if the coefficients c_n of a power series $\sum_0^\infty c_n z^n$ are quadruply monotonic, that is, $\Delta^{(\nu)} c_n \geqq 0$, $n=0, 1, 2, \cdots$; $\nu = 0, 1, \cdots, 4$, where

$$\Delta^{(\nu)} c_n = c_n - \binom{\nu}{1} c_{n+1} + \binom{\nu}{2} c_{n+2} - \cdots + (-1)^\nu \binom{\nu}{\nu} c_{n+\nu},$$

then the series is convergent and univalent for $|z|<1$. The author gives a simple construction, by means of the lemma, of a polynomial which is not univalent in the unit circle and whose coefficients are monotonic of order two.

M. S. Robertson (New Brunswick, N. J.).

Robertson, M. S. Typically-real functions with $a_n = 0$ for $n \equiv 0 \pmod 4$. Bull. Amer. Math. Soc. **46**, 136–141 (1940). [MF 1261]

Let $f(z) = z + \sum_2^\infty a_n z^n$ be regular and typically-real for $|z|<1$; that is, $f(z)$ be real when and only when z is real. If, in addition, $a_n = 0$ for $n \equiv 0 \pmod 4$, the author proves some sharp inequalities for the $|a_n|$; for instance,

$$|a_n| + 2^{-\frac{1}{2}}(n-1)|a_2| \leqq n, \qquad n \text{ odd};$$
$$|a_2| \leqq 2^{\frac{1}{2}}, \quad |a_n| + |a_2| \leqq 2^{\frac{1}{2}}, \qquad n \text{ even}.$$

W. W. Rogosinski (Cambridge, England).

Miyatake, Osamu. On Riemann's ξ-function. Tôhoku Math. J. **46**, 160–172 (1939). [MF 1180]

Der Verfasser formuliert Seite 169 eine Annahme; diese betrifft die absolute Konvergenz einer Reihe, welche in Zusammenhang mit einer in der Riemann'schen Integraldarstellung der ξ-Funktion auftretenden Funktion steht. Aus dieser Annahme zieht der Verfasser durch Überlegungen, die allerdings stellenweise schon im sprachlichen Ausdruck etwas dunkel sind, eine Folgerung über die Nullstellen von $\xi(s)$, also über die nichttrivialen Nullstellen von $\zeta(s)$, welche ein neues, einschneidendes Resultat in der Richtung der Riemann'schen Vermutung wäre. Allerdings zieht der Verfasser schon vorgängig aus derselber Annahme einen Schluss, einen Grenzwert betreffend [Seite 169, $\psi(z) \to 0$], woraus insbesondere sofort folgen würde, dass $\zeta(s)$, in einer gewissen Halbebene, entlang jeder Parallelen zur imaginären Achse für $s \to \infty$ den Grenzwert 1 hat. Da dies jedoch bekanntlich nicht zutrifft, ist entweder die Annahme falsch, oder ist daraus der erwähnte Schluss falsch gezogen worden.

G. Pólya (Zürich).

Zuckerman, Herbert S. On the expansions of certain modular forms of positive dimension. Amer. J. Math. **62**, 127–152 (1940). [MF 964]

Hardy and Ramanujan have expanded modular forms of positive dimension which have a finite number of poles in the fundamental region, but are regular at $i\infty$ [Proc. Roy. Soc. London. Ser. A. **95**, 144–155 (1919); also S. Ramanujan's Collected Papers, pp. 310–321]. The author combines their method with that developed by Rademacher and Zuckerman [Ann. of Math. **39**, 433–462 (1938)] in order to treat the general case of a meromorphic modular form $F(\tau)$ of positive dimension, admitting a pole at $i\infty$. The transformation equations of $F(\tau)$ are given as

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a, b, c, d)\cdot(-i(c\tau+d))^{-r} F(\tau), \qquad c>0,$$
$$F(\tau+1) = e^{2\pi i\alpha} F(\tau), \qquad 0 \leqq \alpha < 1.$$

The function $F(\tau)$ is supposed to have only a finite number of poles in the fundamental region. For $\Im(\tau) > A$, it is regular and admits there of a Fourier expansion

$$(*) \qquad f(e^{2\pi i\tau}) = e^{-2\pi i\alpha\tau} F(\tau) = \sum_{n=-\mu}^{\infty} a_n e^{2\pi i n\tau}.$$

As Hardy and Ramanujan, the author starts with

$$\frac{1}{2\pi i}\int_{C_N} \frac{f(x)}{x-y}dx = f(y) + R(N),$$

where y is a point of regularity and $R(N)$ represents the sum of the residues arising from the poles inside C_N. The path of integration C_N requires some attention. The author does not take that one employed by Hardy and Ramanujan. He chooses a geometrically simpler path, which can best be described in the variable τ after the substitution $x = \exp(2\pi i\tau)$. In the upper τ-halfplane the author considers first the semicircles which are the image of $\Re(\tau)=0$ by all the modular transformations

$$\tau' = \frac{h\tau + h'}{k\tau + k'},$$

where h'/k' and h/k are two adjacent fractions of the Farey series of order N, $0 \leqq h'/k' < h/k \leqq 1$. These semicircles form a continuous chain from 0 to 1. Whereas in the Hardy-Ramanujan paper quoted above the integration can be extended into the cusps h/k of the path, here the cusps have to be cut off by horizontal segments $\Im(\tau) = B_N$, since in the rational points $F(\tau)$ becomes infinite. The constant B_N does not need to be precisely fixed, but $B_N < 1/2N^2$ is necessary in order that all semicircles are hit by $\Im(\tau) = B_N$. Moreover B_N is chosen as so small that the triangular pieces cut off by it contain no poles in the interior. (The author, incidentally, writes simply B instead of B_N.) On this path of integration only those parts yield essential contributions which consist of a horizontal piece and pieces of the semicircles in the neighborhood, that is, only the parts of the path close to the rational points h/k. This decomposition replaces here the customary "Farey-dissection." On each piece of the path of integration the transformation equation is applied. This provides an approximation to $F(\tau)$ in the neighborhood of each rational point. After the necessary estimations and after the passage to the limit $N \to \infty$, the result is

$$f(y) = 2\pi \sum_{k=1}^{\infty} \frac{1}{k} \sum_{\nu=1}^{\mu} a_{-\nu} \sum_{n=0}^{\infty} A_{k,\nu}(n) \left(\frac{\nu-\alpha}{n+\alpha}\right)^{(r+1)/2}$$
$$\times I_{r+1}\left(\frac{4\pi}{k}((\nu-\alpha)(n+\alpha))^{\frac{1}{2}}\right) y^n - R(\infty),$$

where $A_{k,\nu}(n)$ are certain arithmetical sums, the $a_{-\nu}$ stem from the "principal part" of the Fourier series (*), and $R(\infty)$ stands for $\lim_{N\to\infty} R(N)$. The sum of residues $R(\infty)$ is then fully discussed in the case that the fundamental region besides the pole at $i\infty$ contains only one pole σ in its interior. It is interesting to note that the expression shows a modular form in the parameter σ of the negative dimen-

sion $-r-2$, appearing as a Poincaré series of the generalized type studied by Petersson.

In a further paragraph the class of all functions which satisfy the conditions of the main theorem is characterized in terms of $g_2(1,\tau)$, $g_3(1,\tau)$, $\eta(\tau)$, and $J(\tau)$. This parametrization of all those functions furnishes the particularly interesting example

$$F_r(\tau)=\frac{1}{1728}\eta(\tau)^{-24}(J(\tau)-J(\sigma))^{-1}$$

as covered by the main theorem. For $0<r\leqq 12$ the expansion is very simple and can be completely given. A comparison of leading terms yields the formula

$$1728\,\eta(\sigma)^{24}J'(\sigma)=-\frac{3\cdot 13!}{2^{11}\pi^{13}}i\sum_{p=-\infty}^{+\infty}\sum_{q=-\infty}^{+\infty}(p\sigma+q)^{-14},$$

where a special Eisenstein series appears on the right-hand side.

The author finally discusses briefly also the above-mentioned $F_r(\tau)$ for $12<r\leqq 24$. *H. Rademacher.*

Jung, Heinrich W. E. Zur Theorie der algebraischen Funktionen zweier Veränderlicher. III. Über die Zahl δ der Zeuthen-Segreschen Invariante. J. Reine Angew. Math. **181**, 125–132 (1939). [MF 1416]

Let $<\mathfrak{G}>$ be a pencil of genus p consisting of integral divisors of genus q on an algebraic surface K. The author associates to $<\mathfrak{G}>$ a number δ which is closely related to the Zeuthen-Segre invariant of K. It is proved that $\delta\geqq 0$ for certain fields and suitable pencils $<\mathfrak{G}>$: (i) K is neither rational nor semi-rational (direct product of a rational and irrational curve), $<\mathfrak{G}>$ arbitrary; (ii) if K is rational or irrational and $<\mathfrak{G}>$ is irreducible. Reducible pencils may yield $\delta<0$ as is shown, too. The proofs are stated in the divisor theoretic terminology which the author adopted to the classic methods of algebraic geometry. They essentially depend on the author's definition of "place" and "intersection multiplicity" and "transformation of places" which cannot be evaluated here. *O. F. G. Schilling.*

Behnke, H. Über die Fortsetzbarkeit analytischer Funktionen mehrerer Veränderlichen und den Zusammenhang der Singularitäten. Math. Ann. **117**, 89–97 (1939). [MF 1385]

Verfasser beweist, dass die Singularitäten einer ein- oder mehrdeutigen analytischen Funktion $f(z_1, z_2, \cdots, z_n)$ von n komplexen Veränderlichen, deren Regularitätsbereich nur erreichbare Randpunkte hat, eine zusammenhängende Mannigfaltigkeit bilden. (Die Bedingung, dass der Regularitätsbereich nur erreichbare Randpunkte besitzt, ist equivalent mit der, dass er "endlichblättrig" ist.) Dies ist eine Verallgemeinerung eines Satzes von Cacciopoli, der nur unter den einschränkenden Bedingungen, dass die Funktion schlicht und die Gesamtheit ihrer singulären Punkte eine dreidimensionale, zweimal differenzierbare Mannigfaltigkeit bildet, bewiesen wurde [Atti 1° Congr. Unione Mat. Ital. 1938]. Aus dem erstgenannten Satze folgt: Die endlichbrättrige Funktion $f(z_1, z_2, \cdots, z_n)$ sei regulär und eindeutig auf einer geschlossenen, über dem $z_1\cdots z_n$-Raume gelegenen $(2n-1)$-dimensionalen Fläche $\mathfrak{F}$, wobei in einem Punkte P von $\mathfrak{F}$ unter den in ihm zugelassenen Funktionselementen eines eindeutig festgelegt sei. Ist es dann unmöglich die Funktion f von einer Seite von $\mathfrak{F}$ zur andern ohne Überschneidung der auf $\mathfrak{F}$ eindeutig bestimmten Funktionselemente fortzusetzen, so ist f nach einer Seite von $\mathfrak{F}$ unbeschränkt fortsetzbar. Dieser Satz enthält als Spezialfall den bekannten Hartogs'schen Satz, dass eine Funktion $f(z_1, \cdots, z_n)$, die in sämtlichen Randpunkten eines schlichten beschränkten Bereiches $\mathfrak{B}$ mit zusammenhängendem Rande regulär ist, sich ins ganze Innere von $\mathfrak{B}$ eindeutig und regulär fortsetzen lässt. *P. Thullen* (Quito).

Fuchs, B. A. Über eine Eigenschaft der bei pseudokonformen Abbildungen invarianten Metrik. Rec. Math. (Moscou) [Mat. Sbornik] N.S. **5** (47), 497–504 (1939). (Russian. German summary) [MF 1340]

For every univalent domain $\mathfrak{B}^{2n}$ of the space of n complex variables there exists the "kernel-function" $K(z,\bar{z})$, $z=(z^1,\cdots,z^n)$ by means of which a metric $G_{\mathfrak{B}^{2n}}$ invariant with respect to ps.-c.tr. (pseudo-conformal transformations) can be defined: the line-element is $ds^2=\sum T_{m\bar{n}}dz^m d\bar{z}^n$, $T_{m\bar{n}}=\partial^2\log K/\partial z^m\partial\bar{z}^n$. The transformation of $\mathfrak{B}^{2n}$ into the "representative domain" $\mathfrak{R}^{2n}(\mathfrak{B}^{2n},t)$ of $\mathfrak{B}^{2n}$ with respect to t is defined by

$$u^k=\sum T^{\bar{k}\nu}(t)\partial\log M(z,t)/\partial\bar{t}^\nu,\quad M(z,t)=K(z,\bar{t})/K(t,\bar{t})$$

[Bergmann, J. Reine Angew. Math. **169**, 1–40 (1932); Math. Annalen **102**, 430–446 (1930)]. If $n=1$ and $\mathfrak{B}^2$ is simply connected, then $\mathfrak{R}^2(\mathfrak{B}^2,t)$ is a circle with the center in t. If $n\geqq 2$, $\mathfrak{R}^{2n}(\mathfrak{B}^{2n},t)$ possesses certain important properties. If $\mathfrak{B}^2$ is simply connected the bundle of straight lines forms the bundle of the geodesic lines of $G_{\mathfrak{B}^2}$ through t in the circle $\mathfrak{R}^2(\mathfrak{B}^2,t)$. If $n=2$ the bundle T of analytic surfaces $u^1+cu^2=0$, $u_k=\partial\log M(z,t)/\partial\bar{t}^k$ will be transformed into a bundle of analytic planes through t of $\mathfrak{R}^4(\mathfrak{B}^4,t)$. All t.g.a.s. (totally geodesic analytical surfaces) passing through t belong to T. A surface $\mathfrak{S}^2$ is called a t.g.s. if the tensor $H^{\cdot}_{mn}$ of the enforced curvature vanished at every point of $\mathfrak{S}^2$. [Cf. Fuchs, Rec. Math. (Moscou) [Mat. Sbornik] N.S. **2** (44), 567–598 (1937).] The author proves the following theorem: If there exists one point t of $\mathfrak{B}^4$ such that a complete bundle of t.g.a.s. passes through t, then the metric $G_{\mathfrak{B}^4}$ has a constant unitary curvature and there exists a complete bundle of t.g.a.s. at every point of $\mathfrak{B}^4$. If furthermore the domain of regularity of K is $\mathfrak{R}^4(\mathfrak{B}^4,t)$, then $\mathfrak{R}^4(\mathfrak{B}^4,t)$ is a hypersphere in the case considered. ("Complete" bundle means that there exists in every analytic direction a t.g.a.s.) For the proof of this theorem the author uses a previous result: In order that an analytic surface is a t.g.a.s. it must satisfy a certain differential equation. Substituting the expression obtained for T in this equation the author shows that M and therefore the kernel-function K satisfy a certain differential equation [cf. also previous paper of the author, Bull. Inst. Math. Méc. Tomsk **1**, 168–174 (1935)]. He deduces from this result that K has the form $K(u^1,u^2;\bar{u}^1,\bar{u}^2)=|\phi(u^1,u^2)|^2f(u^1\bar{u}^1+u^2\bar{u}^2)$, where $\phi(u^1,u^2)$ is an analytic function of u^1, u^2. The line element ds of $G_{\mathfrak{B}^4}$ is therefore invariant with respect to the group of ps.-c.tr. which leaves $u^1\bar{u}^1+u^2\bar{u}^2$ invariant. Using this fact he obtains the statement of the theorem. Furthermore he proves the following theorem: If a bundle of geodesic lines passing through t forms an analytic surface, then this surface is a t.g.a.s. *S. Bergmann* (Cambridge, Mass.).

Stein, Karl. Über das zweite Cousinsche Problem und die Quotientendarstellung meromorpher Funktionen mehrerer Veränderlichen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1939**, 139–149 (1939). [MF 1565]

Let B be a region in the space of n complex variables $z_1,\cdots,z_n$. To every point P in B let there be associated a neighborhood $U(P)$ and a function $f_P(z_1,\cdots,z_n)$ regular in

$U(P)$. Furthermore, let f_P/f_Q be regular and non-vanishing in the intersection $U(P)\cdot U(Q)$. Cousin's (second) assertion is said to hold in B if whenever the above holds there exists a single-valued function F regular in B and such that F/f_P is regular and non-vanishing in $U(P)$. The determination of regions for which Cousin's assertion is valid has been the object of several investigations. It is known that the assertion holds in some regions and not in others. For example, Cousin himself proved that a cylindrical region, at most one of whose projections is multiply connected, is a region for which his assertion holds. The general question as to when his assertion is or is not valid remains open. In the present paper the author concludes that the difficulties are of a topological nature. He derives a necessary condition (of a topological nature) for the validity of Cousin's assertion in a general region. He then obtains some interesting related results for certain cylindrical regions. Along with these results he obtains similar results, as well as one stronger result, on Poincaré's assertion (Poincaré's assertion holds in B if every single-valued function f meromorphic in B can be written $f=g/h$, where g and h are single-valued and regular in B and prime to each other). The author closes the paper by using these considerations to construct a Schlicht non-Runge regularity-region. *W. T. Martin.*

Vignaux, J. C. Functions of one and of several complex variables on any surface. An. Soc. Ci. Argentina **128**, 3–9 (1939). (Spanish) [MF 1078]

Let the analytic function $f(z)$ be defined on some surface. A theorem on the possibility of an analytic continuation of $f(z)$ is stated, which is a consequence of Cauchy's integral theorem for functions on surfaces. Furthermore, the author considers functions of two variables, each of them varying on a surface, and obtains an analogue to Cauchy's formula by applying the formula for a single function twice.

W. Feller (Providence, R. I.).

Kodama, Sikazô. Sur la classe quasi-analytique de fonctions de deux variables (I, II and III). Mem. Coll. Sci. Kyoto Imp. Univ. Ser. A. **22**, 269–356 (1939). [MF 1495, 1496, 1497]

Following largely the methods of Carleman and to a lesser extent those of Mandelbrojt and Ostrowski, the author extends the known theory of quasi-analytic functions of a single variable to two variables. Classes C_M are defined by the inequalities $|f^{(m+n)}(x, y)|\leqq k^{m+n}M_{m,n}$ $(a\leqq x\leqq A;$ $b\leqq y\leqq B)$ $(m, n=0, 1, \cdots)$, where k may depend on the choice of f. On letting $\beta_{m,n}^{m+n}=M_{m,n}$, $\beta_{m,n}^{*}=\min \beta_{m',n'}$ $(m'+n'\geqq m+n)$, $T(R, r)=\max R^m r^n/M_{m,n}$ $(m, n\geqq 1)$, it is proved that a class C_M is quasi-analytic if and only if

$$\sum\sum \frac{1}{\beta_{m,n}^{*}} \quad\text{or}\quad \int_1^\infty\int_1^\infty \log T(R, r)R^{-2}r^{-2}dRdr$$

diverges. It is shown that, given a quasi-analytic class C_M, constants $\omega_{p,q}^{m,n}$ can be found so that every f, of C_M, is expressible as

$$\lim_{m,n}\sum_{p=0}^{m}\sum_{q=0}^{n}\omega_{p,q}^{m,n}f^{(p+q)}(0, 0)\frac{1}{p!q!}x^p y^q$$

in the region $(0\leqq x\leqq a,\ 0\leqq y\leqq b)$; in order that there should exist a function f of a quasi-analytic class C_M, with $f^{(p+q)}(0, 0)=p!q!c_{p,q}$ $(p, q\geqq 0)$, it is necessary and sufficient that a certain sequence of forms, quadratic in the $c_{p,q}$, be bounded. In the last part of the paper the point of view is that of N. Wiener and the study is given, with the aid of Fourier transforms, for functions f of classes $C_M{}^*$, characterized by inequalities

$$\iint |f^{(m+n)}(x, y)|^2dxdy\leqq k^{m+n}M_{m,n}^2, \qquad m, n\geqq 0.$$

It is proved that divergence of the integral

$$\int_0^\infty\int_0^\infty \log T(x, y)\frac{dxdy}{(1+x^2)(1+y^2)},$$

where $T(x, y)=\max M_{m,n}^{-2}x^{2m}y^{2n}$ $(m, n\geqq 0)$, is necessary and sufficient for quasi-analyticity of the class

$$C_M{}^*\begin{pmatrix}-1, 1\\ -1, 1\end{pmatrix}.$$

W. J. Trjitzinsky (Urbana, Ill.).

Theory of Series

Johnston, L. S. The Fibonacci sequence and allied trigonometric identities. Amer. Math. Monthly **47**, 85–89 (1940). [MF 1389]

The author derives several identities involving inverse circular and hyperbolic cotangents and certain terms of the Fibonacci series:

$$1, 1, 2, 3, 5, 8, \cdots, u_n, \cdots.$$

The following are typical:

$$\operatorname{arc\,cot} u_{2p}=\operatorname{arc\,cot} u_{2p+1}+\operatorname{arc\,cot} u_{2p+2}$$
$$\operatorname{arc\,coth} u_{2p-1}=\operatorname{arc\,coth} u_{2p}+\operatorname{arc\,coth} u_{2p+1}$$
$$\operatorname{arc\,cot} u_{2p-1}+\operatorname{arc\,cot} u_{2p+2}=\operatorname{arc\,cot} (u_{2p}/u_3).$$

D. H. Lehmer (Bethlehem, Pa.).

Orts, J. M.[a] Two notes on numerical series. Revista Mat. Hisp.-Amer. (3) **1**, 29–36 (1939). (Spanish) [MF 1672]

Some series, closely connected with the harmonic series, are discussed by using the asymptotic formula for $\sum_{\nu=1}^{n}1/\nu$.

O. Szász (Cincinnati, Ohio).

Selberg, Sigmund. Über die Reihe für die Eulersche Konstante, die von E. Jacobsthal und V. Brun angegeben ist. Norske Vid. Selsk. Forh. **12**, 89–92 (1939). [MF 1184]

A new proof is given for the following representation of Euler's constant:

$$E=\sum_1^\infty \frac{(-1)^n}{n}\left[\frac{\log n}{\log 2}\right],$$

which was given first by E. Jacobsthal in 1906 and in a slightly different form by V. Brun in 1938.

O. Szász (Cincinnati, Ohio).

Rajagopal, C. T. Some theorems connected with Maclaurin's integral test. Math. Gaz. **23**, 456–461 (1939). [MF 1379]

Several theorems analogous to the Maclaurin (Cauchy) integral test for the convergence of infinite series are proved. These theorems are outgrowths of certain theorems proved by J. E. Littlewood and G. H. Hardy. In addition, a constant analogous to Euler's constant is defined and certain properties of it are developed.

A paper by H. Chand [Amer. Math. Monthly **46**, 338–341 (1939); these Rev. **1**, 10 (1940)] should be added to the bibliography. *T. Fort* (Bethlehem, Pa.).

Schilling, Bernhard. Methoden zur numerischen Auswertung unendlicher Reihen mit reellen konstanten Gliedern. J. Reine Angew. Math. 181, 177–192 (1939). [MF 1419]

The author considers convergent series of real terms which obey certain monotonic conditions. The purpose of the paper is to display a method for obtaining decimal approximations for s, the sum of the series, with less labor than the direct addition of terms of the series. Well-known treatments of the general problem involved are by Euler, Kummer and Markoff. Cf. also papers by J. A. Shohat [Amer. Math. Monthly 40, 226–229 (1933)] and J. W. Bradshaw [Amer. Math. Monthly 46, 486–492 (1939)]. The methods employed in the paper under review are strictly elementary but quite effective. The author sets up upper and lower bounds for R_n, the remainder after n terms, and it is by utilization of these bounds that he achieves his results. Examples treated in detail include

$$\sum_1^\infty \frac{1}{n^2},\quad \sum_1^\infty \frac{(-1)^{n+1}}{2n-1},\quad \sum_1^\infty \frac{1}{n^{1.1}}$$

and

$$\sum_1^\infty \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)}\frac{1}{(2n-1)}.$$

T. Fort (Bethlehem, Pa.).

Bochner, S. A generalization of Poisson's summation formula. Duke Math. J. 6, 229–234 (1940). [MF 1554]

The sum $\sum_{-\infty}^{\infty} f(m)$ occurring in Poisson's summation formula [see S. Bochner, Fouriersche Integrale, 33] can be regarded as the sum of the residues of $\pi f(z) \cot \pi z$, provided that $f(z)$ is analytic in a strip of the complex plane $|y|<y_0$. In the paper under review the function $\cot \pi z$ is replaced by an unspecified meromorphic function with a consequent generalization of Poisson's formula. *T. Fort.*

Erdös, Paul and Turán, Paul. On the uniformly-dense distribution of certain sequences of points. Ann. of Math. 41, 162–173 (1940). [MF 1009]

Let $\{\zeta_\nu^{(n)}\}$ be a triangular sequence of numbers such that $1\geqq\zeta_1^{(n)}>\cdots>\zeta_n^{(n)}\geqq-1$. Let furthermore $\zeta_\nu^{(n)}=\cos\phi_\nu^{(n)}$, $0\leqq\phi_\nu^{(n)}\leqq\pi$, and $\omega_n(\zeta)=\prod_{\nu=1}^n(\zeta-\zeta_\nu^{(n)})$. The authors prove that if the estimate $|\omega_n(\zeta)|<2^{-n}A(n)$ holds in $(-1, 1)$ for every n, then for every subinterval (α, β) of $(0, \pi)$ one has

$$\left|\sum_{\alpha\leqq\varphi_\nu^{(n)}\leqq\beta} 1-\frac{\beta-\alpha}{\pi}n\right|<\frac{8}{\log 3}(n\log A(n))^{\frac12}.$$

M. Kac (Ithaca, N. Y.).

Scott, W. T. and Wall, H. S. A convergence theorem for continued fractions. Trans. Amer. Math. Soc. 47, 155–172 (1940). [MF 1075]

Using strictly elementary methods the authors prove that, if $a_2, a_3, a_4, \cdots$ are complex numbers and if there exist non-negative numbers $r_1, r_2, r_3, \cdots$ such that (1) $r_n|1+a_n+a_{n+1}|\geqq r_nr_{n-2}|a_n|+|a_{n+1}|$ $(n=1, 2, 3, \cdots; r_0=r_{-1}=a_1=0)$, the series $1+\sum r_1r_2\cdots r_n$ dominates the series $|A_1/B_1|+\sum_2^\infty D_n$, where $D_n=|A_n/B_n-A_{n-1}/B_{n-1}|$ and where A_n/B_n is the nth convergent of the continued fraction

$$(2)\qquad \frac{1}{1+}\frac{a_2}{1+}\frac{a_3}{1+}\cdots.$$

The continued fraction then converges if some $a_n=0$, or if the series $1+\sum r_1r_2\cdots r_n$ converges. These results are employed to provide proofs for well-known theorems of Worpitzky and of Van Vleck, and to improve notably a theorem of Pringsheim [see Perron, Die Lehre von den Kettenbrüchen, 2nd ed., pp. 258–264]. The similarity of conditions (1) to conditions of Leighton [Duke Math. J. 4, 775 (1938)] is investigated and the authors find a proof along similar lines of a somewhat strengthened form of his theorem.

The most striking result of the paper is the "parabola" theorem which states that if the elements a_n of (2) are not 0 and lie in or on the parabola $|z|-\Re(z)=1/2$, (2) converges if and only if the series $\sum|b_n|$ diverges, where $b_1=1$ and $a_n=1/b_{n-1}b_n$ $(n=2, 3, 4, \cdots)$. Further, this parabola is the best curve symmetric in the real axis having these properties.

W. Leighton (Houston, Tex.).

Schwartz, H. M. A class of continued fractions. Duke Math. J. 6, 48–65 (1940). [MF 1542]

Let $P_n(z)/Q_n(z)$ be the nth approximant of the continued fraction

$$\mathop{K}_{i=1}^{\infty}\left(\frac{k_i}{z+c_i}\right),$$

where $k_i\neq 0$, c_i are complex numbers, and z is a complex variable. The writer shows that, if $\sum|k_i|$ converges, then the sequences

$$\{P_n(z)/\prod_{i=1}^n(z-c_i)\},\quad \{Q_n(z)/\prod_{i=1}^n(z-c_i)\}$$

converge uniformly over every domain of the z-plane at a positive distance from the set $\{c_i\}$. If $\sum|c_i^{-1}|$ and $\sum|k_{i+1}/c_ic_{i+1}|$ converge, then the sequences

$$\{P_n(z)/\prod_{i=1}^n(-c_i)\},\quad \{Q_n(z)/\prod_{i=1}^n(-c_i)\}$$

converge uniformly over every bounded region to entire limit-functions. The special case where $\lim c_i$ exists is considered. The theorems generalize well-known results for the continued fraction

$$\mathop{K}_{i=1}^{\infty}(k_iz/1),$$

and are obtained by methods similar to those used by Maillet [J. École Polytéchn. (2) 12, 41–62 (1908)] in the case of the latter continued fraction. As an example the continued fraction of Bessel is treated. *H. S. Wall.*

Izumi, Shin-ichi and Kawata, Tatsuo. On certain series of functions. Tôhoku Math. J. 46, 91–105 (1939). [MF 1172]

Let $\phi(t)$ be such that

$$(1)\ \phi(t+1)=\phi(t),\quad (2)\ \int_0^1\phi(t)dt=0,\quad (3)\ \int_0^1\phi^4(t)dt<\infty,$$

and

$$(4)\qquad \int_0^1|\phi(t+h)-\phi(t)|^4dt<A|h|^{4\alpha},$$

for every h and some α $(0<\alpha\leqq1)$. Furthermore, let $\{n_k\}$ be an increasing sequence of integers for which (*) $\sum k^{\frac12}/2^{\alpha(n_{k+1}-n_k)/2}<\infty$. The authors then prove that, if $\sum_1^\infty a_k^2<\infty$, the series $\sum_1^\infty a_k\phi(2^{n_k}t)$ converges almost everywhere. Moreover, if (*) is replaced by $\lim_{k\to\infty}k/2^{\alpha(n_{k+1}-n_k)/2}=0$, and the other conditions remain unaltered, the convergence almost everywhere of $\sum_1^\infty a_k\phi(2^{n_k}t)$ implies $\sum_1^\infty a_k^2<\infty$. The paper refers to earlier work on similar questions by Rademacher [Math. Ann. 87, 112–138 (1922)], Khintchine and

Kolmogoroff [Rec. Math. 32, 668–677 (1928)] and the reviewer [Studia Math. 7, 96–100 (1938); J. London Math. Soc. 12, 131–134 (1937)]. *M. Kac* (Ithaca, N. Y.).

Barone, Henry G. Limit points of sequences and their transforms by methods of summability. Duke Math. J. 5, 740–752 (1939). [MF 815]

Let $\sigma_n=\sum_{k=1}^{n} a_{nk}s_k$ be the transform of a sequence s_n by the method of summability determined by the matrix a_{nk}. The main problem of the paper is that of determining whether a given matrix has the following property P: the set of limit points of the transform σ_n of each bounded complex sequence s_n is a connected point set in the complex plane. Some criteria are given which ensure that a_{nk} has property P; other criteria are given which ensure that a_{nk} does not have the property. In particular, the problem is solved for Hölder, Cesàro, Riesz, de la Vallée Poussin, and Euler methods of various real and complex orders.

R. P. Agnew (Ithaca, N. Y.).

Meyer-König, Werner. Über einige Sätze aus der Reihenlehre. Math. Z. 45, 751–755 (1939). [MF 1412]

If $\{a_n\}$ converges to zero by a linear transformation satisfying a permanence-condition, then $\sum_{\nu=0}^{n} a_\nu - na_n \to s$ implies $\sum_{n=0}^{\infty} a_n = s$. *S. Izumi* (Sendai).

Ríos, S. On the domains of convergence of the algorithms of convergence (E_n) which generalize that of Euler. Revista Mat. Hisp.-Amer. (3) 1, 37–44 (1939). (Spanish) [MF 1673]

This paper is a sequel to a previous paper [Rev. Mat. Hisp.-Amer. 37 (1932)] where a comparatively simple generalization of Euler summability [Knopp, Unendliche Reihen, 2. Aufl., 509] is applied to the power series

$$1+z+z^2+\cdots. \tag{1}$$

Let D_n denote the domain of summability of (1) of order n. The object of the paper under consideration is to prove that, if $m\neq n$, D_n never contains all points of D_m.

T. Fort (Bethlehem, Pa.).

Cesco, R. P. On the theory of linear transformations and the summation of divergent series. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Revista (2) 2, no. 127, 156–169 (1940). (Spanish) [MF 1727]

Let $A=\|a_{n,\lambda}\|$ be a matrix of real or complex numbers. Let S_λ be the partial sum of the series $\sum u_\mu$. Let $\mathfrak{A}$ be the set of all sequences $\{S_\lambda\}$ such that $\lim_{n\to\infty}\sum_{\lambda=0}^{\infty} a_{n,\lambda}S_\lambda$ exists. Let $\mathfrak{A}^*$ be the partial set of $\mathfrak{A}$ such that, for every $\{S_\lambda\}\epsilon\mathfrak{A}^*$, the following relation takes place:

$$\lim_{n\to\infty}\sum_{\mu=0}^{\infty} a_{n,\lambda+\mu}u_\mu=0, \qquad \lambda=0, 1, 2, \cdots.$$

$\mathfrak{A}^*$ is called the range of analyticity of A; and A is said to be totally analytic if $\mathfrak{A}^*=\mathfrak{A}$. The author, after showing that all regular Nörlund matrices N_p ($p\geqq 0$) are totally analytic, establishes necessary and sufficient conditions in order that a matrix, the elements of whose main diagonal are different from zero, be totally analytic. He then shows that certain matrices considered by Agnew [Ann. of Math. 32, 715 (1931)] and by Raff [Math. Z. 36, 1 (1933)] are not totally analytic. Finally he proves three theorems on the A-summability of the product series of two series. Let us quote the following: Let V_λ and U_λ be the partial sums of $\sum v_\mu$ and $\sum u_\mu=U$, and let the last series converge absolutely. Let W_λ be the partial sum of the product series, and let $A=\|a_{n\lambda}\|$. Then the conditions (a) $\{V_\lambda\}$ belongs to the range of analyticity of A; (b) $|a_{n,\lambda}V_0+a_{n,\lambda+1}V_1+\cdots+a_{n,n}V_{n-\lambda}|<M$, $n=0, 1, 2, \cdots$; $\lambda=0, 1, 2, \cdots, n$, are necessary and sufficient in order that

$$\lim_{n\to\infty}\sum_{\lambda=0}^{\infty} a_{n,\lambda}W_\lambda=UV.$$

A. González Domínguez (Providence, R. I.).

Amerio, Luigi. Sulle condizioni di validità dei metodi di sommazione di Gronwall. Ann. Mat. Pura Appl. 18, 239–259 (1939). [MF 921]

Given $f(w)$ analytic for $|w|\leqq 1$ except at $w=1$, and such that $z=f(w)$ maps $|w|<1$ simply on a region D interior to $|z|<1$ in such fashion that $z=0$ corresponds to $w=0$ and $z=1$ to $w=1$. Furthermore, the inverse function is analytic on the boundary of D, except perhaps at $z=1$, where we have $1-w=(1-z)^\lambda\psi(z)$, $\lambda\geqq 1$, $\psi(z)$ being analytic in D and on its boundary and $\psi(1)>0$. Given also

$$g(w)=\sum_{n=0}^{\infty} b_nw^n=(1-w)^{-\alpha}+\gamma(w), \quad b_n\neq 0 \text{ for all } n;\ \alpha>0,$$

where $\gamma(w)$ is analytic for $|w|\leqq 1$ and $g(w)\neq 0$ for $|w|<1$. If $U_0, U_1, U_2, \cdots$ are generated by the identity

$$\sum_{m=0}^{\infty} u_mz^m=\frac{1}{g(w)}\sum_{n=0}^{\infty} b_nU_nw^n,$$

and $U_n\to s$ as $n\to\infty$, the series $\sum u_n$ is said to be summable (f, g) to the sum s. This type of definition was introduced by Gronwall in 1932 [Ann. of Math. (2) 33, 101–117 (1932)] and was shown to include de la Vallée Poussin's method for proper choice of f and g. In the present paper the properties of such methods of summation are studied under less restriction on the functions f and g than in Gronwall's original discussion. The main application made is to the theory of analytic extension of a power series. *C. N. Moore.*

Hyslop, J. M. Some theorems on absolute Cesàro summability. Proc. Edinburgh Math. Soc. (2) 6, 114–122 (1939). [MF 1525]

The paper is closely related to another paper [Proc. Edinburgh Math. Soc. 5, 182–201 (1938)] by the same author. Let $c_n^{(k)}$ denote the kth Cesàro mean of the series $\sum a_n$, and let $a_n^{(k)}=c_n^{(k)}-c_{n-1}^{(k)}$. The chief result of the paper may be stated as follows. Let p be any positive integer and let $0<\rho<1$. A necessary and sufficient condition that the series $\sum n^\rho a_n$ should be absolutely summable by Cesàro's method of order p is that the series $\sum n^\rho|a_n^{(p)}|$ should be convergent. The theorem must be slightly modified in the case $\rho=1$. *A. Zygmund* (Cambridge, Mass.).

Garten, V. Über die Beziehungen zwischen den Hölderschen und Laplace-Abelschen Mittelbildungen und dem Satz von O. Hölder. Math. Z. 46, 86–103 (1940). [MF 1483]

A direct proof is given of a classic theorem of Hölder which states that if a sequence is summable to s by the Hölder method of some positive integer order, then the sequence is summable to s by the Abel power series method. Inequalities are obtained which give relations between the superior limits of the Hölder, Cesàro and Abel transforms of a real sequence of nonnegative elements. Finally, inequalities are given which relate the inferior and superior limits and oscillations of Hölder integral transforms $h^{(k)}(y)$ of a function $s(x)$ to those of the Abel (or Laplace) integral trans-

form $a(y)$ of the function. Let $\Omega(H_k)$ and $\Omega(A)$ denote the oscillations of $h^{(k)}(y)$ and $a(y)$. The "best constant" γ_k such that $\Omega(H_k)\leqq\gamma_k\Omega(A)$ is greater than 1 when $k=2, 3, 4, \cdots$. Determination of the "best constant" could be facilitated by use of a paper of the reviewer [Properties of generalized definitions of limit, Bull. Amer. Math. Soc. **45**, 689–730 (1939); these Rev. **1**, 50–51 (1940)]. *R. P. Agnew.*

Garten, V. Ungleichungen zwischen den Hauptlimites der von Herrn Karamata untersuchten iterierten Mittelbildungen bei drei aufeinanderfolgenden Ordnungen. Math. Z. **45**, 735–746 (1939). [MF 1410]

Let $0=P_0<P_1\leqq P_2\leqq P_3\leqq\cdots$. If $s(t)$ is integrable over each finite interval $0\leqq t\leqq x$, then $\sigma^{(0)}(t)=s(t)$ and the recursion formulas

$$\sigma^{(k)}(x)=P_kx^{-P_k}\int_0^x t^{P_k-1}\sigma^{(k-1)}(t)dt$$

define transforms of $s(t)$ which become, for special choices of $P_1, P_2, \cdots$, Hölder and Cesàro transforms of nonnegative integer orders. Let s_k and S_k denote the inferior and superior limits of $\sigma^{(k)}(x)$ as $x\rightarrow\infty$. The inequality

$$s_{k-1}\leqq t_k\leqq s_k\leqq s_{k+1}\leqq S_{k+1}\leqq S_k\leqq T_k\leqq S_{k-1},$$

in which t_k and T_k are functions of s_{k-1}, S_{k-1}, s_{k+1} and S_{k+1} analogous to functions of Winn [Sur l'oscillation des moyennes de Hölder et de Cesàro, C. R. Acad. Sci. Paris **194**, 1057–1060 (1932)], is established and used to obtain further results. If $S_k\neq s_k$ and $S_{k-1}=S_k$, then $S_{k+1}=S_k$. If $s_{k-1}\geqq 0$ and $P_{k+1}=(1+\alpha)P_k$, then (i) $S_k\leqq(1+\alpha)^{1/\alpha}S_{k+1}$ provided $0<\alpha\leqq 1$, (ii) $S_k\leqq eS_{k+1}$ provided $\alpha=0$, and (iii) $s_k\geqq\alpha(1+\alpha)^{-1}s_{k+1}$ provided $\alpha\geqq 0$. *R. P. Agnew.*

Karamata, J. Über die Indexverschiebung beim Borelschen Limitierungsverfahren. Math. Z. **45**, 635–641 (1939). [MF 1405]

Let B denote the Borel exponential method of summability by which a sequence $s_0, s_1, \cdots$ is summable B to s if $e^{-x}\sum_{n=0}^{\infty}(x^n/n!)s_n\rightarrow s$ as $x\rightarrow\infty$. It is shown that if $s_0, s_1, s_2, \cdots$ is summable B to s, then $s_1, s_2, \cdots$ will be summable B to s if (i) $s_n=O(c_n)$, where $c_n=\exp n^\rho$ and $\rho<1/3$. The hypothesis (i) may be replaced by the weaker hypothesis that the points $c_n^{-1}s_n$ all lie in a sector in the complex plane with vertical angle less than π. The method of proof fails when $\rho=1/3$. *R. P. Agnew* (Ithaca, N. Y.).

Garabedian, H. L. Theorems associated with the Riesz and the Dirichlet's series methods of summation. Bull. Amer. Math. Soc. **45**, 891–895 (1939). [MF 777]

If ν_n is a logarithmico-exponential function of λ_n satisfying suitable conditions, then each series $\sum u_n$ summable by the Riesz "discontinuous" method of order 1 determined by the sequence λ_n is summable to the same value by the Dirichlet series method determined by the sequence ν_n. A similar theorem applies to series $\sum u_n$ whose partial sums are such that $\lim_{n\to\infty} s_ne^{-\nu_n\sigma}=0$ when $\sigma>0$. *R. P. Agnew* (Ithaca, N. Y.).

Meyer-König, Werner. Zur Frage der Umkehrung des C- und A-Verfahrens bei Doppelfolgen. Math. Z. **46**, 157–160 (1940). [MF 1487]

An example is given of a divergent double series $\sum a_{mn}$ such that mna_{mn} is bounded and converges to 0 as $m, n\rightarrow\infty$, while $\sum a_{mn}$ is summable to 0 by the C_1 and Abel methods of summability. This example answers a question raised by Knopp [these Rev. **1**, 51 (1940)]. Another example and further results are contained in a paper by the reviewer which is in course of publication in the American Journal of Mathematics. *R. P. Agnew* (Ithaca, N. Y.).

Hill, J. D. On perfect summability of double sequences. Bull. Amer. Math. Soc. **46**, 327–331 (1940). [MF 1842]

A linear method of summation, defined by a matrix $A=\{a_{ik}\}$, is called perfect if (a) it transforms every convergent sequence into a sequence convergent to the same limit, (b) if every system of equations $a_{i1}x_1+a_{i2}x_2+\cdots=\eta_i$ $(i=1, 2, \cdots)$ has a unique solution provided $\lim\eta_i$ exists, (c) this solution is identically 0, provided that $\sum|x_i|<\infty$. Banach proved [Théorie des opérations linéaires, p. 95] that, if A is a perfect method and B is a method satisfying condition (a) and not weaker than A, then every sequence summable A is also summable B to the same limit. The present paper gives an extension of this result to the case of double sequences. The statement is more complicated than in the case of single sequences. *A. Zygmund.*

Kloosterman, H. D. Tauberian theorems for Cesaro-summability of double series. Nederl. Akad. Wetensch., Proc. **43**, 215–223 (1940). [MF 1594]

The author proves Tauberian theorems in double series for Cesàro-summability of any order. Let

$$\sum_{m,n=1}^{\infty} a_{m,n}$$

be a double series. Let $s_{m,n}^{(-1,-1)}=a_{m,n}$. If t and r are integers not less than -1, let

$$s_{m,n}^{(t+1,r)}=\sum_{\mu=1}^{m} s_{\mu,n}^{(t,r)},\quad s_{(m,n)}^{(t,r+1)}=\sum_{\nu=1}^{n} s_{m,\nu}^{(t,r)},\quad m, n=1, 2, \cdots.$$

For t and $r\geqq 0$, the double series is called summable $(C; t, r)$ if

$$\lim_{m,n\to\infty}\frac{s_{m,n}^{(t,r)}}{\binom{m+t-1}{t}\binom{n+r-1}{r}}$$

exists. The Tauberian theorems are then proved under the condition

$$s_{m,n}^{(0,-1)}<\frac{K}{n},\quad s_{m,n}^{(-1,0)}<\frac{K}{m},\quad m, n=1, 2, \cdots,$$

for constant K, or under the alternative condition

$$a_{m,n}<\frac{K}{m^2+n^2},\qquad m, n=1, 2, \cdots.$$

N. Levinson (Cambridge, Mass.).

TOPOLOGY

Markoff, A. On the definition of a complex. Rec. Math. (Moscou) [Mat. Sbornik] N.S. **5** (47), 545–550 (1939). (English. Russian summary) [MF 1345]

This presents a simplified version of a finite Euclidean simplicial complex [cf. Alexandroff-Hopf, Topologie] in terms of which it is easy to establish a theorem on small displacements of such complexes. The author takes a finite set of points in Euclidean n-space as a "geometric simplex" rather than the convex envelope of that set. *A. W. Tucker* (Princeton, N. J.).

Whitney, Hassler. Some combinatorial properties of complexes. Proc. Nat. Acad. Sci. U. S. A. **26**, 143–148 (1940). [MF 1285]

This note supplements the author's paper "On products in a complex" [Ann. of Math. **39**, 397–432 (1938)] to meet the needs of the note reviewed below. It sets up his cup product in terms of the intersections of one "dual system" with another in general position. A "dual system" is a bilinear chain-function $D(A, B)$ such that $\dim B - \dim A = \dim D(A, B)$ and $D(\delta A, B) - D(A, \delta B) = (-1)^n \delta D(A, B)$, where $n = 1 + \dim A$. For a simplicial complex, $D(s, t)$ may conveniently be taken as that chain in a regular subdivision which is carried by the transverse of the simplex s relative to the closure of the simplex t [in the sense of Lefschetz, Topology, p. 117]. The second part of the note extends the author's product theory to non-augmentable complexes, that is, ones whose vertices cannot be oriented so that their sum is a cocycle. In a way these dualize non-orientable circuits [Topology, p. 48]. To make this extension the author studies abstract complexes, suitably articulated, which are "locally isomorphic," that is, whose cells correspond in a one-one manner so that corresponding cell-closures are isomorphic. These locally isomorphic complexes are sorted into isomorphic types by means of a characteristic one-dimensional cohomology class.

A. W. Tucker (Princeton, N. J.).

Whitney, Hassler. On the theory of sphere-bundles. Proc. Nat. Acad. Sci. U. S. A. **26**, 148–153 (1940). [MF 1286]

This note sketches the author's recent findings on sphere-bundles, previously called sphere-spaces. The account is enigmatically brief and intricate, but the author promises treatment in book form later. A sphere-bundle is a space X which may be mapped on another space Y so that the inverse image of a point y of Y is a subset $S(y)$ of X which is homeomorphic to a sphere S_0 of given dimension. Thus X is pictured as a "bundle" of spherical "fibres" $S(y)$, one "over" each point y of the "base" space Y. It is further prescribed that for each element V of a suitably chosen covering of Y by open sets and each point y of V there is a homeomorphism $h(y, V)$ of S_0 on the fibre $S(y)$ such that (1) $h(y, V)$ varies continuously as y varies in V and (2) if y belongs to the common part of V_1 and V_2, then $h(y, V_1)$ and $h^{-1}(y, V_2)$ combine to produce an orthogonal transformation of S_0 on itself. (Replacing S_0 by a general space and the orthogonal group by some group of self-homeomorphisms of that space one gets the more general notion of fibre-bundle.) The author uses the combinatorial devices developed in the note reviewed above to get new results on sphere-bundles. To quote from his introduction he presents "further properties of the characteristic classes, a duality theorem, theorems on tangent and normal bundles to a manifold, and some examples."

A. W. Tucker.

Friedgé, Hans. Verallgemeinerung der Dodekaederräume. Math. Z. **46**, 27–44 (1940). [MF 1475]

A class of 3-dimensional manifolds $\mathfrak{M}^{(n)}$ is formed by taking as fundamental region a polyhedron $\mathfrak{P}^{(n)}$ having as faces two n-agons separated by two rings of pentagons, as in the dodecahedron ($=\mathfrak{P}^{(5)}$). The matching of faces follows the method for the "spherical dodecahedron-space," which is $\mathfrak{M}^{(5)}$ [see Seifert and Threlfall, Topologie, 216]. The fundamental group is calculated. Except when $n \equiv 3 \bmod 6$ it is generated by two elements A and B with the following relations ($\epsilon = \pm 1$): if $n = 6k$, $[A, [A, B]^{k-2}] = [B, [A, B]^{k-2}] = 1$; if $n = 6k + \epsilon$, $(AB)^2 = B^3 = A^{6+\epsilon}(B^{-3}A^6)^{k-\epsilon}$; if $n = 6k + 2\epsilon$, $A^3B^3 = (AB^2)^{2k+\epsilon}A^{3k} = 1$. When $n = 6k + 3$ there are three generators with the relations $A_1^2 = A_2^{-2} = A_3^2$, $(A_1A_2A_3)^{2k+1} = A_1^{2k}$. [The second relation for $n = 6k - 2$ is given by the author in a form which does not reveal the symmetry with $6k + 2$. The form here given is obtained from his by noting that from $A^3B^3 = 1$ it follows (1) that A^3 and AB^2 commute, and (2) that $AB^{-1}A^{-2}B^{-1} = A^3(AB^2)^2$.] If $n \equiv \pm 1 \bmod 6$, $\mathfrak{M}^{(n)}$ is a Poincaré space, and, by applying Seifert's necessary and sufficient condition in terms of "Fasern" [Seifert, Acta Math. **60**, 211 (1932)], it is shown that every $\mathfrak{M}^{(6k\pm1)}$ is one of the spaces constructed by Dehn's method from a clover-leaf knot. Other manifolds $\mathfrak{M}_r^{(n)}$ are then considered, formed by different matchings of the faces of $\mathfrak{P}^{(n)}$. They are shown to include spaces with the same homology groups as the lens spaces, but different fundamental groups.

M. H. A. Newman (Cambridge, England).

Choquet, Gustave. Homéomorphies. C. R. Acad. Sci. Paris **210**, 129–131 (1940). [MF 1289]

The author here continues his investigation [for earlier results, see C. R. Acad. Sci. Paris **206**, 634–636 (1938)] of closed plane sets H having the property that every homeomorphism of H into another plane set can be extended to the entire plane. In case a continuum C is the sum of a Jordan continuum and the homeomorphic image of one or more half lines with their points of accumulation on the boundary of the Jordan continuum, necessary and sufficient conditions that C have the above property are found. The problem is studied also for sets which are not connected.

D. Montgomery (Northampton, Mass.).

Swingle, P. M. A finitely-containing connected set. Bull. Amer. Math. Soc. **46**, 178–181 (1940). [MF 1267]

An example is given of a connected set which, for every $n \geqq 2$, is the sum of n mutually exclusive biconnected sets but is not the sum of (in fact does not contain) infinitely many mutually exclusive connected sets containing more than one point. The construction is carried out in the plane using the hypothesis of the continuum and Zermelo's axiom. It follows closely the methods used by E. W. Miller to obtain a biconnected set without a dispersion point [Fund. Math. **29**, 123–133 (1937)].

W. L. Ayres.

Milgram, Arthur N. Partially ordered sets and topology. Proc. Nat. Acad. Sci. U. S. A. **26**, 291–293 (1940). [MF 1809]

The author gives conditions on a partially ordered set P necessary and sufficient for the existence of a topological space having a basis of closed sets isomorphic to P (this space need not however be unique). He does this by defining "separating systems" of any P. Using this definition, he also exhibits the abstract essence of a reduction theorem of Brouwer and a dual covering theorem of Borel on completely separable spaces.

G. Birkhoff (Princeton, N. J.).

Dieudonné, J. Sur les espaces uniformes complets. Ann. École Norm. **56**, 277–291 (1939). [MF 1860]

A space possessing a uniform structure may or may not [examples here and in Dieudonné, C. R. Acad. Sci. Paris **209**, 145–147 (1939); these Rev. **1**, 30 (1940)] possess a uniform structure in which it is complete. The class of spaces possessing such a uniform structure coincides with the class of spaces homeomorphic to a closed subset of a product of metrizable spaces. Such a space need not be normal.

J. W. Tukey (Princeton, N. J.).

Novák, Josef. Sur les espaces (*L*) et sur les produits cartésiens (*L*). Publ. Fac. Sci. Univ. Masaryk 1939, no. 273, 28 pp. (1939). [MF 1015]

Each (*L*)-space generates in an obvious way a neighborhood-space satisfying Hausdorff's axioms (*A*) and (*B*) (but not necessarily the remaining axioms) so that the (*L*)-spaces can be regarded as a particular case of the general topological spaces. A detailed and systematic study of the (*L*)-spaces from the point of view of their "neighborhood-properties" is given, with a resulting classification of the (*L*)-spaces and examples demonstrating various possibilities (for example, a (*L*)-space in which no two points can be separated by open sets, a bicompact (*L*)-space which does not satisfy the first countability axiom, etc.). In particular, the author investigates the separation properties of the (*L*)-spaces and solves a problem of Fréchet concerning necessary and sufficient conditions for the validity of the separation-postulates T_1, T_2, T_3, T_4. The second part of the paper is dedicated to the investigation of the Cartesian products of (*L*)-spaces. *W. Hurewicz.*

Adkisson, V. W. and MacLane, Saunders. Extending maps of plane Peano continua. Duke Math. J. 6, 216–228 (1940). [MF 1553]

The authors consider the problem of finding conditions under which a homeomorphism T of a Peanian continuum M on a sphere S and a continuum M' on a sphere S' can be extended to a homeomorphism T' of the entire sphere S into S'. Gehman solved this problem for Peanian continua immersed in the plane by showing that a necessary and sufficient condition is that T preserve sides of arcs. The authors point out that Gehman's conditions can easily be extended to the sphere. The main theorem of the paper furnishes a necessary and sufficient condition for the sphere from a new point of view, namely that the transformation is required to preserve the relative sense of triods, where a triod is a set of three arcs, mutually exclusive except for a common end point. In the plane there must be added the condition that the boundary of the unbounded complementary domain of M goes into the boundary of the unbounded complementary domain of M'. The second main theorem reduces the above conditions somewhat by showing that it is not necessary to assume that relative sense of all triods be preserved but that it suffices to assume that this holds for a certain specified subset of the set of all triods of M and M'. *J. R. Kline* (Philadelphia, Pa.).

Hall, Dick Wick. On a decomposition of true cyclic elements. Trans. Amer. Math. Soc. 47, 305–321 (1940). [MF 1587]

The paper gives the first attempt at a set theoretic decomposition of the cyclic elements of a Peano space into smaller elements, which are called secondary elements. The complicated theory which results gives many results analogous to the cyclic element theory. Some interesting examples are given at the close of the paper which show the imperfection (and difficulty) of the theory in that the secondary elements here defined may be both connected and locally connected, may possess one property without the other, or may possess neither. *W. L. Ayres* (Ann Arbor, Mich.).

Youngs, J. W. T. *K*-cyclic elements. Amer. J. Math. 62, 449–456 (1940). [MF 1776]

Let E be a cyclically connected Peano space. A point a is said to be k-conjugate to a point b if, for any k distinct points $x_1, \cdots, x_k$ in E, distinct from a and b, a and b are in the same component of $E-x_1-x_2-\cdots-x_k$; if $k=2$, a is said to be bi-conjugate to b. A (nondegenerate) k-cyclic (bi-cyclic if $k=2$) element is the totality of points each of which is k-conjugate to each of $k+1$ distinct points which are k-conjugate in pairs. The following facts are proved concerning k-cyclic elements in general: Any two points of a k-cyclic element are k-conjugate to one another and there is at most one k-cyclic element containing $k+1$ distinct points. If two k-cyclic elements have k points in common, then these k points separate E. Each k-cyclic element is closed and there are at most a denumerable number of them in E. In fact, the limit inferior of any sequence of distinct k-cyclic elements can contain at most k distinct points. Finally, every nondegenerate continuum of convergence is contained in some k-cyclic element for each k.

The following further facts concerning bi-cyclic elements are proved: If M is a bi-cyclic element, the frontier of a component S of $E-M$ consists of exactly two points which are in M and which together separate E; any other bi-cyclic element N of E is either entirely in the closure $\bar{S}$ of some component S of $E-M$ or has no points in common with S. For each $\delta>0$, there are only a finite number of bi-cyclic elements of diameter greater than δ. Finally, if $\{K_n\}$ is a sequence of continua such that (1) $L=\lim K_n$ contains at least three points, (2) $L\cdot\sum_{n=1}^{\infty}K_n=0$, then there exists a bi-cyclic element M such that $L=\lim [K_n\cdot M]$. *C. B. Morrey* (Berkeley, Calif.).

Miller, Edwin W. Some theorems on continua. Bull. Amer. Math. Soc. 46, 150–157 (1940). [MF 1263]

The paper contains the following extension of the well-known theorem of Anna Mullikin: If C is a compact metric continuum and F_1 and F_2 are mutually exclusive non-vacuous closed subsets of C, then $C-(F_1+F_2)$ contains a constituent C_1 such that $\bar{C}_1\cdot F_1\neq 0\neq \bar{C}_1\cdot F_2$. The present extension replaces the word "component" of the Mullikin theorem by "constituent." The proof involves the use of the Mullikin theorem. The stronger theorem gives the following applications to plane domains: (a) If the bounded continuum B is the common boundary of at least two domains and C is a nondegenerate closed proper subset of B, then some constituent of $B-C$ has at least two limit points in C. (b) If the continua M_n are mutually exclusive, no one cuts the plane, and each lies in a bounded domain D except possibly for one point, then $M=\sum M_n$ does not cut D.

If $M=\sum M_n$, where the sets M_n are mutually exclusive closed subsets of a compact metric space, it is known [Sierpiński, Tohôku Math. J. 13, 300–303 (1918)] that M contains no M-join, that is, a continuum intersecting at least two of the sets M_n. The paper contains two conditions under which $M+P$ contains no M-join, where P is closed and totally disconnected. It proves also that if $M+N_k$ contains no M-join, N_k closed, then $M+\sum N_k$ contains no M-join if (a) each N_k is totally disconnected or (b) the sets N_k are mutually exclusive and each intersects at most one set M_n. The proofs involve the strengthened Mullikin result. *W. L. Ayres* (Ann Arbor, Mich.).

Parhomenko, A. Über eineindeutige stetige Abbildungen. Rec. Math. (Moscou) [Mat. Sbornik] N.S. 5 (47), 197–210 (1939). (Russian. German summary) [MF 1430]

Necessary and sufficient conditions are given for the existence of a one-to-one and continuous mapping of a topological space onto (1a) a metric space, (1b) a symmetric

space, (2) a zero-dimensional Hausdorff space, (3) a zero-dimensional metric space with denumerable basis, (4) a zero-dimensional compactum. The conditions in cases (1a), (3) and (4) concern the existence of certain types of sequences of open coverings of the topological space. The condition for case (1b) is that every point be the intersection of a denumerable number of its neighborhoods. For case (2) the condition is that for any two points of the topological space, x_1 and x_2, there should be a division of the space into two open, disjoint sets, one containing x_1 and the other x_2. *J. V. Wehausen* (New York, N. Y.).

Fox, Ralph H. On homotopy and extension of mappings. Proc. Nat. Acad. Sci. U. S. A. **26**, 26–28 (1940). [1000]

The author states theorems on homotopy properties of mappings, particularly of mappings of sets into absolute neighborhood retracts. Relations with the "category" of a set, and with particular deformations, "expansions," are considered. *H. Whitney* (Cambridge, Mass.).

Eilenberg, Samuel. Cohomology and continuous mappings. Ann. of Math. **41**, 231–251 (1940). [MF 1013]

Let K be an arbitrary (finite or infinite) geometrical cell complex and f a continuous mapping of K in a topological space Y. Let σ^{n+1} be a $(n+1)$-cell of K. The mapping f considered on the boundary of σ^{n+1} defines an element $c(f, \sigma^{n+1})$ of the nth homotopy group $\pi_n(Y)$ of Y. By taking $c(f, \sigma^{n+1})$ as the coefficient of σ^{n+1}, we obtain an $(n+1)$-chain $c^{n+1}(f)$ which turns out to be a cocycle. Chains with elements of homotopy groups as coefficients were first used by H. Whitney [Duke Math. J. **3**, 50 (1937)], who also was first to use cohomologies instead of homologies in the formulation and proof of the fundamental result of H. Hopf on mappings of complexes in spheres. The author shows that the cohomology properties of the cocycle $c^{n+1}(f)$ defined above are closely related to the homotopy properties and extension possibilities of the mapping f. His theory applied to the case of combinatorial manifolds contains the results he established in a previous paper [Fund. Math. **31**, 179–200 (1938)], where cycles instead of cocycles were used. The author studies, in particular, the case when the homotopy groups $\pi_i(Y)$ vanish for $i=1, 2, \cdots, n-1$ and gives a generalization of Hopf's theorem which includes the generalizations of H. Whitney and the reviewer. *W. Hurewicz.*

Kelley, J. L. A decomposition of compact continua and related theorems on fixed sets under continuous transformations. Proc. Nat. Acad. Sci. U. S. A. **26**, 192–194 (1940). [MF 1602]

This is a digest of results contained in a paper offered by the author in June, 1939, to Fundamenta Mathematicae. A decomposition of (compact) continua into sets, called F-sets, is given which parallels the decomposition of locally connected continua into cyclic elements. Similar theorems are obtained; notably, it is shown that every point of a continuum is either a cut point, an end point, or a point of some nondegenerate F-set. Furthermore, unicoherence is F-set extensible. Applications to the study of invariant sets under continuous transformations are made. If $T(M) \subset M$ is continuous, where M is a continuum, it is shown that (1) there exists a continuum π contained in some F-set in M such that $T(\pi) \supset \pi$; (2) there exists either a fixed point or an F-set F in M such that $F \cdot T(F)$ contains a nondegenerate continuum; (3) there exists a compact subset R of an F-set of M such that $T(R)=R$; (4) if each F-set maps into a subset of an F-set, there exists an F-set F such that $T(F) \subset F$. *G. T. Whyburn* (Charlottesville, Va.).

Wallace, A. D. Monotone coverings and monotone transformations. Duke Math. J. **6**, 31–37 (1940). [MF 1540]

The paper relates monotonic transformations to monotonic coverings of the space, that is, coverings with connected sets. In order that a Peano space be unicoherent it is necessary and sufficient that (a) every one-dimensional monotonic image be acyclic, or (b) every one-dimensional monotonic covering be acyclic. The dimension of the covering is the largest number n such that $n+1$ sets of the coverings intersect. As the author informs the editors, this theorem as well as theorem 1 of the paper are immediate consequences of results of S. Eilenberg [Fund. Math. **29**, 109–110 (1937); **27**, 174 (1936)]. The theorem is applied to obtain a new proof of Eilenberg's theorem that in Peano spaces unicoherence is preserved under interior transformations. A covering $A=\sum A_\alpha$ is said to be free if there exists a continuous transformation f of A into itself such that $A_\alpha \cdot f(A_\alpha)=0$ for every α. A transformation $T(A)=B$ is free if the covering of A with the sets $T^{-1}(x)$ is a free covering. Hopf [Fund. Math. **28** (1937)] has shown that no free transformation carrying a continuum into an arc exists. This result is extended to give: No free monotonic transformation carrying a continuum into a dendrite exists. The familiar Scherrer fixed-point theorem is obtained as a corollary. *W. L. Ayres* (Ann Arbor, Mich.).

Wallace, A. D. On 0-regular transformations. Amer. J. Math. **62**, 277–284 (1940). [MF 1763]

This paper defines and develops some of the properties of a new transformation. A continuous transformation $T(A)=B$ is defined to be 0-regular if for any sequence of points y_n converging to y in B, the sets $T^{-1}(y_n)$ converge to $T^{-1}(y)$ 0-regularly, that is, for any $\epsilon>0$ there exist δ and N such that for $n>N$ any two points u and v of $T^{-1}(y_n)$ with $\rho(u, v)<\delta$ lie in an ϵ-continuum of $T^{-1}(y_n)$. It follows immediately from the characterization of Eilenberg that any 0-regular transformation is interior. A transformation on a compact space is 0-regular if and only if, for any sequence of closed sets Y_n converging to Y 0-regularly, the inverse sets $T^{-1}(Y_n)$ converge to $T^{-1}(Y)$ 0-regularly. The most striking result is that any 0-regular transformation on a continuum can be factored $T=T_2T_1$, so that T_1 is 0-regular and monotonic and T_2 is a local homeomorphism of constant multiplicity. The product of two 0-regular transformations is 0-regular. For Peano continua it is shown that 0-regular transformations preserve cut points and arc-sets. The following minor theorem (3.4) of the paper appears to be incorrect: If the point x of A is an end point, a regular point in the sense of Menger or a cut point of A, then x is a component of $T^{-1}T(x)$ and T is locally topological in a neighborhood of x. The statement becomes correct if the portion from the word "and" on is deleted and this portion is not involved in any of the other results of the paper. *W. L. Ayres* (Ann Arbor, Mich.).

Puckett, W. T., Jr. On 0-regular surface transformations. Trans. Amer. Math. Soc. **47**, 95–113 (1940). [MF 1072]

A continuous transformation $T(M)=M'$ is 0-regular provided that, if (x_i') is a sequence in M' converging to a point x' in M', then $\{T^{-1}(x_i')\}$ converges to $T^{-1}(x')$, and for any $\epsilon>0$ there exist numbers $\delta, N>0$ such that, if $i>N$ and $x, y \epsilon T^{-1}(x_i')$, $\rho(x, y)<\delta$, x and y lie together in a continuum

in $T^{-1}(x_i')$ of diameter less than ϵ. In this paper a study is made of such transformations acting on 2-dimensional pseudo-manifolds. It is shown that if M is such a manifold and $T(M)=M'$ is 0-regular and monotone, either T is topological or M' is an arc or a simple closed curve. Furthermore, T must be topological or M' must reduce to a single point except in the following cases: (1) the sphere, 2-cell and circular ring may be mapped onto an arc; (2) the torus Klein bottle, circular ring, Möbius band, pinched sphere, and 2-cell with two boundary points identified may be mapped onto a simple closed curve. In each of these cases all possible transformations of the sort considered are completely characterized. For example, the only non-topological 0-regular monotone transformation of a sphere onto a nondegenerate image is topologically equivalent to an orthogonal projection onto a diameter. The relation of these results to retracting transformations and to equi-continuous collections of curves is brought out and the general 0-regular transformation is discussed and the possible images determined by the method of factorization into monotone and light factors. *G. T. Whyburn* (Charlottesville, Va.).

Puckett, W. T., Jr. Regular transformations. Duke Math. J. 6, 80–88 (1940). [MF 1544]

A continuous mapping $T(M)=M'$ of a continuum M is called r-regular if (a) T is interior and (b) the sets $T^{-1}(a')$ for $a'\epsilon M'$ are "equally" locally-connected in the homology sense in the dimensions $0, 1, \cdots, r$. If T is r-regular and $s=0, 1, \cdots, r+1$, then the subgroup H^s of the s-dimensional Betti group $B^s(M)$ determined by the cycles of $T^{-1}(a')$ is independent of the choice of $a'\epsilon M'$. If T is 0-regular, then the factor group $B^1(M)/H^1$ is isomorphic with $B^1(M^1)$. *W. W. S. Claytor* (Washington, D. C.).

Vance, E. P. Generalizations of non-alternating and non-separating transformations. Duke Math. J. 6, 66–79 (1940). [MF 1543]

This paper introduces and develops a modification of the previously defined non-alternating, non-separating, completely non-alternating and completely non-separating transformations [G. T. Whyburn, Amer. J. Math. **56**, 294–302 (1934) and J. F. Wardwell, Duke Math. J. **2**, 745–750 (1936)]. The earlier definitions were so strong as to give little information on dendrites or dendritic portions of the space. For example, it was impossible to define a completely non-separating transformation on a dendrite. The author defines a transformation $T(A)=B$ to be weakly non-separating if every $T^{-1}(x)$, $x\epsilon B$, which separates two points u and v in A, contains a single point which separates u and v. The other three transformations are similarly weakened. The characteristic properties and product and factor theorems are developed. At the close of the paper, transformations which are locally non-separating and non-alternating are defined and studied. The most striking result here is that, for any compact metric continuum A, $T(A)$ is a Peano space for any locally non-separating T. *W. L. Ayres.*

Harrold, O. G., Jr. The non-existence of a certain type of continuous transformation. Duke Math. J. 5, 789–793 (1939). [MF 819]

The author proves that there does not exist a continuous transformation defined on an arc A such that each point in the image space has exactly 2 inverse points in A. He remarks that for $k=1, 3, 4, 5, \cdots$ it is possible to define an exactly $(k, 1)$-continuous transformation on an arc. *W. Hurewicz* (Chapel Hill, N. C.).

MATHEMATICAL PHYSICS

Sobrero, Luigi. Sopra un problema di elettrostatica. Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 10, 143–157 (1939). [MF 1019]

This paper gives a new derivation of the formulae for the coefficients of induction q_{11}, q_{12}, q_{22} of two conducting spheres. The method seems to have no particular advantages over that generally used [e.g., in Jeans' Mathematical Theory of Electricity and Magnetism, 1915, 196–199]. *E. T. Copson* (Dundee).

Adams, E. P. Note on a problem in electrostatics. Quart. J. Math., Oxford Ser. 10, 241–246 (1939). [MF 1033]

The solution of two-dimensional electrical and hydrodynamical problems connected with a grating of rounded bars was obtained by H. W. Richmond [Proc. London Math. Soc. (2) **22** (1929)] beginning with a grating of bars of rectangular section. The present paper extends this method to the case where the bars are mid-way between parallel planes. The consequent numerical calculations are performed with the aid of the reviewer's tables of elliptic functions. *L. M. Milne-Thomson* (Greenwich).

Jaeger, J. C. Magnetic screening by hollow circular cylinders. Philos. Mag. 29, 18–31 (1940). [MF 1157]

An infinite, hollow, circular, conducting cylinder is free from electric and magnetic fields. At the instant $t=0$ (i) a uniform magnetic field, (ii) an impulsive field, (iii) a periodical field or (iv) an exponentially damped field is established outside the cylinder and (a) parallel or (b) perpendicular to its axis. To find the field in the hollow interior, in these eight cases, author starts with Maxwell's equations neglecting displacement currents. He integrates these equations by applying the Laplace transformation with respect to t and thus reducing the partial differential equations to ordinary ones. The actual solutions are obtained by inverting the Laplace transformation and evaluating the integrals by the calculus of residues. A few numerical examples, the results of which are shown in diagrams, show the usefulness of the series obtained thus. *A. Erdélyi* (Edinburgh).

Bellustin, S. V. On the currents in vacuo between co-axial cylinders. Acad. Sci. U.S.S.R. J. Phys. 1, 251–262 (1939). [MF 989]

The author assumes two infinitely long coaxial cylinders, being equipotential surfaces. All electrons have equal velocities on coaxial cylindrical surfaces directed radially. The velocities are small with respect to the velocity of light and the phenomena are of a stationary kind. He proposes to calculate the current density, the potential distribution and the space charge density, the electron transit time between coaxial cylinders and the current-voltage characteristics between said surfaces. Four fundamental assumptions are announced regarding possible distributions of space charge density and of potential and of electric field strength. The fundamental differential equation of the problem is shown to be

$$\frac{d}{dR}\left(R\frac{d\Phi}{dR}\right)=\frac{1}{\Phi^{\frac{1}{2}}}.$$

By means of a transformation of S. Lie this equation is rewritten in the form

$$\frac{dw}{du}=\frac{w^2-2u^2-4uw+2u}{3u}.$$

Applying the methods of H. Poincaré the singular points, limiting cycles and separatrissae of the latter equation are calculated and discussed by means of illustrating graphs. Some numerical results are compared with figures given by I. Langmuir. In particular, the quantity β of this author is discussed by the present method. Several possible distributions of potential are discussed and shown by graphs. The calculated current-voltage curves shown in graphs contain the well-known double valued regions which are peculiar to problems of this kind. *M. J. O. Strutt* (Eindhoven).

Ferraro, V. C. A. The induction of currents in infinite plane current-sheets. I. Proc. London Math. Soc. (2) **46**, 99–112 (1940). [MF 1453]

The differential equations of the electromagnetic field of a magnetic system moving with constant velocity in the presence of an imperfectly conducting plane sheet are derived and their complete solution is given. Without loss of generality, the magnetic system can be represented by a single pole. The (approximate) solution given in Maxwell's classical treatise is restricted to vertical motion, that is, cannot give the electrical polarization of the current sheet, and to velocities small compared to that of light, being based on the nonrelativistic transformation. The problem is reduced to three differential equations. The first pertains to a three dimensional potential function B independent of time when related to the moving system; the equation is of first order with the magnetic potential as perturbation function; the components of the induced vector potential are given by the derivatives of B. The other equations, also of first order, connect the components due to magnetic induction and those due to the current sheet; they pertain to the vector and scalar potentials, respectively, and are based on the boundary conditions on the sheet. These equations remain valid for the case of non-uniform motion, except that the Lorentz transformation can no longer be applied to the contravariant four-vector of the potential. The equations must be changed, however, when treating the current sheet from the point of the electron theory rather than from the phenomenological (Maxwellian) point. An approximative solution of that case is to be given in a forthcoming paper, together with applications to the theories of geomagnetic storms and of superconductivity.

H. G. Baerwald (Cleveland, Ohio).

Husimi, Kôdi. On the asymptotic distribution of frequencies of a Hohlraum and the surface tension of an ideal gas. Proc. Phys.-Math. Soc. Japan **21**, 759–768 (1939). [MF 1132]

The familiar expression of Rayleigh for the number of eigenoscillations with frequencies less than ν of a cavity, namely

$$j(\nu)\cong V\cdot\frac{4\pi}{3}\left(\frac{\nu}{c}\right)^3=j_0(\nu)$$

(V being the volume of the cavity and c the velocity of the waves), is of an asymptotic nature only, that is, $\lim_{\nu\to\infty}(j(\nu)/j_0(\nu))=1$, but the difference $j(\nu)-j_0(\nu)$ may be arbitrarily great. The order of magnitude of this difference is usually given as proportional to ν^2. It is the purpose of the author to determine this second order approximation to $j(\nu)$ and to consider its physical effects. The calculation is based on a rectangular cavity with sides a, b, c, and consists in the determination of the number of lattice points with integral coordinates p_1, p_2, p_3 within the ellipsoid

$$\frac{p_1^2}{a_1^2}+\frac{p_2^2}{a_2^2}+\frac{p_3^2}{a_3^2}=\left(\frac{2\nu}{c}\right)^2,$$

the method of generating functions of Darwin and Fowler being applied. A consideration of the one dimensional problem clearly illustrates this method, which consists in the use of the Mellin inversion formula where, in the language of the operational calculus, the partition function is the operational image of the above function $j(\nu)$, so that the latter can at once be written down as a complex integral with an infinite path parallel to the imaginary axis. The partition function appears to be a θ-function. In order to obtain a series for $j(\nu)$ with the Rayleigh approximation as its first term, the argument of the θ-function is inverted by means of the well-known functional relation $\theta(x)=x^{-\frac{1}{2}}\theta(1/x)$, the operational original of the series of this new θ-function leading in the one dimensional case to the step function

$$j_i(\nu)=\left[\frac{2a_i\nu_i}{c}\right]+\begin{cases}1\\0\end{cases}$$

for the two boundary conditions $\partial u/\partial n=0$ and $u=0$, respectively. In an analogous way the three dimensional problem leads to

$$j(\nu)=a_1a_2a_3\frac{4\pi}{3}\left(\frac{\nu}{c}\right)^3\pm\frac{\pi}{2}(a_1a_2+a_2a_3+a_1a_3)\left(\frac{\nu}{c}\right)^2$$
$$+\frac{1}{4}(a_1+a_2+a_3)\frac{2\nu}{c}\pm\frac{1}{8}+\cdots;$$

the terms not written down are of the order unity and fluctuate about the mean value zero. As pointed out by the author in a previous paper [Proc. Phys.-Math. Soc. Japan **20**, 377 (1938)] the appropriate smoothing is obtained if one discards all contributions to the complex integral from regions far from the real axis, a result in accordance with that of Maa. Then the above result is written as

$$(1)\qquad j(\nu)\cong V\frac{4\pi}{3}\left(\frac{\nu}{c}\right)^3\pm\frac{1}{4}A\pi\left(\frac{\nu}{c}\right)^2+\cdots,$$

where V is the volume and A the surface of the rectangular cavity. It is conjectured that this latter result can be extended to a cavity of any shape of volume V and surface A.

When mixed boundary conditions are present, the second term in (1) is hypothetically extended to $\theta(A/4)\pi(\nu/c)^2$, where $-1\leqq\theta\leqq+1$. The physical interest of (1) resides in the fact that even an ideal gas shows under circumstances a surface effect, the two or more terms indicating a phase separation, where the second term leads to a surface tension caused by the kinetic energy of the molecules, whereas the ordinary surface tension is caused by the potential energy. This surface energy becomes of the same order of magnitude as the volume energy for a very thin box of thickness of the order of some ten Ångströms. In conclusion the author remarks that the new surface term requires no modification of London's theory of Einstein-Bose condensation.

B. van der Pol (Eindhoven).

AUTHOR INDEX

PRINTED IN THE UNITED STATES OF AMERICA